Wave function of the universe: a short note

The quantum description of the universe is still an open problem!

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Chapter I

Classical problems

I.I Introduction

Let us consider the equation

$$\ddot{x} = f(x, \dot{x}, t). \tag{I.I}$$

If we know x, $\dot{x}$, and f with all its derivatives at some arbitrary time t_0, we can construct the solution of the equation. Consider

$$\dddot{x} = \partial_x f \dot{x} + \partial_{\dot{x}} f f + \partial_t f. \tag{I.2}$$

The right hand side corresponds to a function that depends on x, $\dot{x}$ and t. This occur for all the derivatives of x and therefore we can write

$$x(t) = x_0 + \dot{x}_0(t - t_0) + \frac{\ddot{x}_0}{2!}(t - t_0)^2 + \frac{\dddot{x}_0}{3!}(t - t_0)^3 + \dots. \tag{I.3}$$

Depending on f, the sum may converge for $|t - t_0| < r$. Recall that if $r \to \infty$, then the series converges for all values of t. In practice, even if the solution is valid for all t, we are interested in $t \geq t_0$ and this type of problem is known as an initial value problem.

As an example, consider

$$f(x, \dot{x}, t) = -\omega^2 x, \tag{I.4}$$

with x_0 and $\dot{x}_0$ given. Then we have

$$\begin{aligned} x(t) &= x_0 + \dot{x}_0(t - t_0) - \frac{x_0\omega^2}{2!}(t - t_0)^2 - \frac{\dot{x}_0\omega^2}{3!}(t - t_0)^3 + \dots \\ &= x_0\left(1 - \frac{\omega^2}{2!}(t - t_0)^2 + \dots\right) + \frac{\dot{x}_0}{\omega}\left(\omega(t - t_0) - \frac{\omega^3}{3!}(t - t_0)^3 + \dots\right), \\ &= x_0 \cos(\omega(t - t_0)) + \frac{\dot{x}_0}{\omega}\sin(\omega(t - t_0)), \end{aligned} \tag{I.5}$$

for all values of t. Of course we could arrive to the same solution by directly solving the differential equation, i.e., the family of solutions are given by

$$x(t) = A\cos(\omega t) + B\sin(\omega t). \tag{1.6}$$

From

$$x_0 = A\cos(\omega t_0) + B\sin(\omega t_0), \quad \dot{x}_0 = -A\omega\sin(\omega t_0) + B\omega\cos(\omega t_0), \tag{1.7}$$

we can write

$$\begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix} = \begin{pmatrix} \cos(\omega t_0) & \sin(\omega t_0) \\ -\omega\sin(\omega t_0) & \omega\cos(\omega t_0) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \tag{1.8}$$

to find

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \cos(\omega t_0)x_0 - \sin(\omega t_0)\frac{\dot{x}_0}{\omega} \\ \sin(\omega t_0)x_0 + \cos(\omega t_0)\frac{\dot{x}_0}{\omega} \end{pmatrix}. \tag{1.9}$$

With these expressions we obtain the result given in equation (1.5). There are other ways to determine A and B. Assume that we give $x_1 = x(t_1)$ and $x_2 = x(t_2)$ with $t_2 > t_1$. Then we will have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(\omega t_1) & \sin(\omega t_1) \\ \cos(\omega t_2) & \sin(\omega t_2) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \tag{1.10}$$

and obtain

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{\sin(\omega t_2)}{\sin(\omega(t_2-t_1))}x_1 - \frac{\sin(\omega t_1)}{\sin(\omega(t_2-t_1))}x_2 \\ -\frac{\cos(\omega t_2)}{\sin(\omega(t_2-t_1))}x_1 + \frac{\cos(\omega t_1)}{\sin(\omega(t_2-t_1))}x_2 \end{pmatrix}. \tag{1.11}$$

This problem is not longer an initial value problem. Since it is set on the interval $t_1 \leq t \leq t_2$ and we have data on the boundaries of the interval, it is called a boundary value problem. In particular, the boundary data is $x(t_1)$ and $x(t_2)$ and thus the problem is called Dirichlet boundary value problem.[1]

For a initial value problem $\ddot{x} = f(x, \dot{x}, t)$ we can consider the first order system

$$y = \dot{x}, \quad \dot{y} = f(x, y, t). \tag{1.12}$$

A state of the system is defined as the point (x, y) in a plane. Thus, the initial value problem corresponds to specify the initial state (x_0, y_0) and to predict a final state of the system at some $t > t_0$. The answer to this question will give a curve in such a plane that joins the initial and final state. Of course, the coordinates of the curve $\Gamma(t) = (x(t), y(t))$ are a solution of the above equations.

For the boundary problem discussed above, we consider the state as the point (x, t) in another plane. Since we know the initial and final state, the question we can

[1] We could instead have $\dot{x}(t_1)$ and $\dot{x}(t_2)$ and this problem is referred as a Neumann boundary value problem.

ask in this case is the curve that join this two points. Again, the curve $\gamma(t) = (x(t), t)$ is a solution of $\ddot{x} = f(x, \dot{x}, t)$. This Dirichlet boundary problem have the property that it can be cast into a variational problem with an action defined as

$$S[x(t)] = \int_{t_1}^{t_2} \mathrm{d}t' \, L(x, \mathrm{d}x/\mathrm{d}t', t'), \tag{1.13}$$

where L is the Lagrangian. The action is a functional and its domain corresponds to the set of curves in the (x, t) plane that joins the initial and final state. Hamilton's principle states that demanding

$$\frac{\delta S}{\delta x(t)} = 0, \tag{1.14}$$

with Dirichlet boundary conditions, i.e. $\delta x(t_1) = 0 = \delta x(t_2)$, is equivalent to demand

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}. \tag{1.15}$$

This is the Euler-Lagrange equation. For a Lagrangian of the form

$$L = \frac{1}{2}\dot{x}^2 - V(x, \dot{x}, t), \tag{1.16}$$

we obtain

$$\ddot{x} = -\frac{\partial V}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial \dot{x}}. \tag{1.17}$$

Thus, the boundary problem can be derived from a variational problem if there is a V such that

$$f(x, \dot{x}, t) = -\frac{\partial V}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial \dot{x}}. \tag{1.18}$$

We will assume that this is possible. Hamilton's principle implies that among all possible curves connecting the initial and final state, then, the actually curve is the one that extremizes the action, i.e. the one that solves the Euler-Lagrange equation.

So far, the plane (x, t) is an abstract space and the action has no geometrical meaning. Moreover, we can generalized the Dirichlet boundary problem by considering $\mathbf{x} = (x^1, \ldots, x^n)$ and thus the curves are defined on $\mathbb{R}^{n+1}$. The line segment in this space is given by

$$\mathrm{d}s^2 = v_0^2 \mathrm{d}t_E^2 + \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j, \tag{1.19}$$

where v_0 has dimensions of velocity such that $\mathrm{d}s^2$ has dimensions of length squared. In a minute a justification of writing t_E instead of t will be given. The length of the curve is $\ell = \int \mathrm{d}s$ and we conviniently parametrized it using t_E and thus

$$\ell = \int_{t_{E1}}^{t_{E2}} \mathrm{d}t_E \left|\frac{\mathrm{d}s}{\mathrm{d}t_E}\right| = v_0 \int_{t_{E1}}^{t_{E2}} \mathrm{d}t_E \sqrt{1 + \frac{1}{v_0^2}\delta_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t_E}\frac{\mathrm{d}x^j}{\mathrm{d}t_E}} \tag{1.20}$$

We see that there is a Dirichlet boundary problem in which the action has a geometrical nature. On the physical perspective, it seems that this scenario has no meaning. This is not true after considering Minkowski space $\mathbb{R}^{n,1}$, i.e. special relativity, rather than its euclidean version $\mathbb{R}^{n+1}$. For this reason, from we used the euclidean time t_E instead of t and thus we see that $v_0 = c$.

We continue in $\mathbb{R}^{n+1}$ but now we further generalize the Dirichlet boundary problem. For $n = 2$ we can consider the problem of finding the surface that joins two different circles, see figure I.I. The states are know the closed curves instead of points

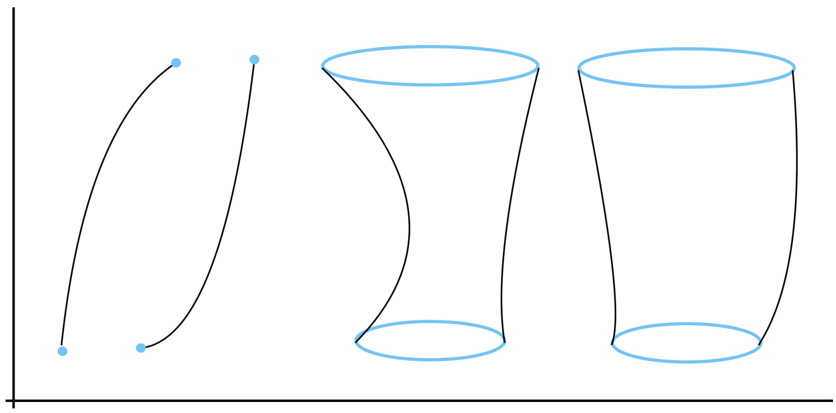

Figure I.I: Illustration of the possible curves/surfaces for the Dirichlet boundary problem.

and the action corresponds the area. For $n = 3$ we consider the states to be two different half spheres and now the action corresponds to a volume. Thus, for general n, the action corresponds to the n-dimensional volume that joins the $(n-1)$-dimensional area states. This generalization finds home in (euclidean) string theory.

The generalization is mathematically pleasing. Notice that the setup is based on the fact that $\mathbb{R}^{n+1}$ is given. The n-dimensional volume and $(n-1)$-dimensional area are imbedded in $\mathbb{R}^{n+1}$. Let us write the line segment of $\mathbb{R}^{n+1}$ as

$$\mathrm{d}s^2_{\mathbb{R}^{n+1}} = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu, \quad g_{\mu\nu} = \delta_{\mu\nu} \tag{I.21}$$

with $\mu, \nu = 0, 1, 2, \ldots n$ and $x^0 = ct_E$. The line segment on the volume, described with coordinates $\{\sigma^i\}$, corresponds to

$$\mathrm{d}s^2_{V^n} = h_{ij}(\sigma)\mathrm{d}\sigma^i \mathrm{d}\sigma^j. \tag{I.22}$$

The metric elements $g_{\mu\nu}$ and $h_{ij}(\sigma)$ are related by

$$h_{ij}(\sigma) = \frac{\partial x^\mu}{\partial \sigma^i}\frac{\partial x^\nu}{\partial \sigma^j} g_{\mu\nu}. \tag{I.23}$$

For the initial and final state, described with coordinates $\{\xi_1^a\}$,$\{\xi_2^a\}$ respectively, we can have

$$\mathrm{d}s^2_{\Sigma_{1,2}} = \gamma_{ab}(\xi_i)\mathrm{d}\xi^a_{1,2}\mathrm{d}\xi^b_{1,2}, \quad \gamma_{ab}(\xi_{1,2}) = \frac{\partial\sigma^i}{\partial\xi^a_{1,2}}\frac{\partial\sigma^j}{\partial\xi^b_{1,2}}h_{ij}(\sigma). \tag{I.24}$$

The action is given by

$$S[x(\sigma)] = \int \mathrm{d}^n\sigma\sqrt{\det h} = \int \mathrm{d}^n\sigma\sqrt{\det\left(\frac{\partial x^\mu}{\partial\sigma^i}\frac{\partial x^\nu}{\partial\sigma^j}g_{\mu\nu}\right)}. \tag{I.25}$$

With Dirichlet boundary conditions we find that the Euler-Lagrange equations are $-\Delta_V x^\mu = 0$. Let us work a $n = 2$ example. We parametrized the surface as

$$t_E = \sigma^1, \quad x^1 = r(\sigma^1)\cos\sigma^2, \quad x^2 = r(\sigma^1)\sin\sigma^2, \tag{I.26}$$

and call $\sigma^2 = \theta$ with $\theta \sim \theta + 2\pi$. Thus we obtain

$$\mathrm{d}s^2_{V^2} = (1+\dot{r}^2)\mathrm{d}t_E^2 + r^2(t_E)\mathrm{d}\theta^2, \tag{I.27}$$

and

$$S = 2\pi\int_{t_{E_1}}^{t_{E2}} \mathrm{d}t_E\, r(t_E)\sqrt{1+\dot{r}^2}. \tag{I.28}$$

The equation to solve is $r\ddot{r} = 1+\dot{r}^2$. A solution is $r(t_E) = r_0\cosh(t_E/r_0)$ and therefore the initial and final states are circles with radius $r(t_{E1,E2}) = r_0\cosh(t_{E1,E2}/r_0)$, see figure I.2.

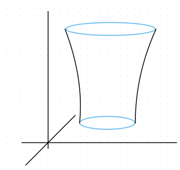

Figure I.2: Example for a $n = 2$ case.

I.2 The action for the universe

A more abstract problem is to consider a boundary problem for $g_{\mu\nu}(x)$ itself. From know on we will work with real time. Let $g_{\mu\nu}(x)$ be the metric elements of a pseudo-Riemannian manifold $\mathcal{M}^{n,1}$, the line segment is

$$\mathrm{d}s^2_{\mathcal{M}^{n,1}} = g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\mu, \tag{I.29}$$

and the equations to satisfy are Einstein's equations

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) + g_{\mu\nu}\Lambda = 8\pi G_N^{(n)} T_{\mu\nu}, \tag{1.30}$$

where $R_{\mu\nu}$ is the Ricci tensor, R the Ricci scalar, Λ the cosmological constant, $T_{\mu\nu}$ the energy-momentum tensor and $G_N^{(n)}$ Newton's constant in n spatial dimensions. Solving these equations is not a simple task. Even the initial value problem is not straight forward. The first step is to assume that the topology of $\mathbb{R} \times \Sigma$, this corresponds to a $n+1$ decomposition and the formalism is referred as to the ADM[2]. Without giving too much details, for the initial time t_0 we must give the (spatial) metric and the extrinsic curvature associated to Σ_{t_0}, specify $T_{\mu\nu}$ and satisfy two constraints (later they will referred as to the Hamiltonian and momentum constraints). Later, there will be two first order in time partial differential equations for the spatial metric and extrinsic curvature. After solving these equations, the result must corresponds to a $n+1$-dimensional metric of a pseudo-Riemannian manifold foliated with hypersurfaces Σ_t for each t.

Another way to solve, which is the way that usually we find solutions, is to make an ansatz of the metric $g_{\mu\nu}$ and $T_{\mu\nu}$. Their form is usually motivated by symmetries. The simplest example is flat spacetime, i.e. Minkowski. Let $g_{\mu\nu} = \eta_{\mu\nu}$ and $T_{\mu\nu} = 0$. One finds that $R_{\mu\nu}(\eta) = 0$ and $R(\eta) = 0$. This imply that $\Lambda = 0$.

Since we are interest in cosmology, consider the following toy cosmological model that satisfy

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) + g_{\mu\nu}\Lambda = 0, \tag{1.31}$$

with

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right). \tag{1.32}$$

Notice that the scale factor $a(t)$ is dimensionless and $1/k$ of length squared. These options indicate that space slices can be hyperbolic, flat, spherical. Also notice that under rescaling of length, i.e. $r \to \lambda r$ where λ, we have $k \to \lambda^{-2}k$ and the metric is invariant if $a \to \lambda^{-1}a$. Notice that $\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2$ correspond to a spatial maximally symmetric space.

The 00 and ij components of Einstein's equations give

$$\left(\frac{\dot a}{a}\right)^2 = \frac{\Lambda}{3} - \frac{k}{a^2}, \quad 2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 = \Lambda - \frac{k}{a^2}, \tag{1.33}$$

respectively. Recall that $1/\Lambda$ has dimensions length squared. By plugging the first equation to the second equation the system becomes

$$\left(\frac{\dot a}{a}\right)^2 = \frac{\Lambda}{3} - \frac{k}{a^2}, \quad \frac{\ddot a}{a} = \frac{\Lambda}{3}. \tag{1.34}$$

[2]For details of this formalism see arXiv:gr-qc/0703035v1.

Notice that the first expressions acts as a constraint. Let us focus on the solutions with $\Lambda > 0$. One possible solution is

$$a(t) = \cosh(\sqrt{\Lambda/3}t), \tag{1.35}$$

where we have used the length rescaling to set the coefficient to unity. Then

$$\tanh^2(\sqrt{\Lambda/3}t) = 1 - \frac{3k}{\Lambda}\,\mathrm{sech}^2(\sqrt{\Lambda/3}t). \tag{1.36}$$

This immediately imply that $k = \Lambda/3 > 0$. If instead we consider

$$a(t) = \sinh(\sqrt{\Lambda/3}t), \tag{1.37}$$

we obtain

$$\coth^2(\sqrt{\Lambda/3}t) = 1 - \frac{3k}{\Lambda}\,\mathrm{csch}^2(\sqrt{\Lambda/3}t), \tag{1.38}$$

and therefore $k = -\Lambda/3 < 0$. Lastly, for $k = 0$ we have $\dot{a} = \pm\sqrt{\Lambda/3}\,a$ and thus we have two options

$$a_\pm(t) = \mathrm{e}^{\pm\sqrt{\Lambda/3}t}. \tag{1.39}$$

By setting

$$T^\Lambda_{\mu\nu} = -\frac{\Lambda}{8\pi G_N^{(3)}}g_{\mu\nu}, \tag{1.40}$$

we see that Λ plays a role of a source. If the above solutions are models of universes, they correspond to the ones in which we are not in them! Nevertheless, we can interpret the solutions for the scenario in which other sources are small compared to $T^\Lambda_{\mu\nu}$.

For these theoretical universes, we see that in the case $a(t) = \cosh(\sqrt{\Lambda/3}t)$ there is no curvature singularity. For $a(t) = \sinh(\sqrt{\Lambda/3}t)$ there is an apparent singularity at $t = 0$. For the remaining cases, the apparent singularities are at $t \to \mp\infty$.

For the singularity free universe, we have

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \cosh^2(\sqrt{\Lambda/3}t)\left(\frac{\mathrm{d}r^2}{1-(\Lambda/3)r^2} + r^2\mathrm{d}\Omega^2\right). \tag{1.41}$$

Let $\Lambda = 3/\ell^2$ and $r = \ell\sin\psi$. We obtain

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \ell^2\cosh^2(t/\ell)\mathrm{d}\Omega_3^2 \tag{1.42}$$

which corresponds to de Sitter spacetime, dS^4, in global coordinates. dS^{n+1}, also its relative AdS^{n+1}, have the property that it can be taught as a hypersurface in an ambient space. For de Sitter, the ambient space is $\mathbb{R}^{n+1,1}$. The hypersurface corresponds to

$$\eta_{AB}X^AX^B = \ell^2, \quad A, B = 0, 1, \dots, n+1. \tag{1.43}$$

Parametrizing the hypersurface as

$$X^0 = \ell \sinh(t/\ell), \quad X^a = \ell \Omega^a \cosh(t/\ell), \tag{1.44}$$

with $t \in \mathbb{R}$ we find

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \ell^2 \cosh^2(t/\ell) \mathrm{d}\Omega_n^2. \tag{1.45}$$

This choice of parametrization gives the global coordinates of dS^{n+1}. A cartoon of de Sitter spacetime is shown in figure 1.3. The solutions $a(t) = \sinh(t/\ell)$ and

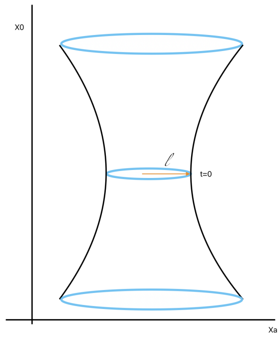

Figure 1.3: de Sitter as an hypersurface.

$a(t) = \mathrm{e}^{\pm t/\ell}$ corresponds to other parametrizations of the hypersurface. The difference between these cases with $a(t) = \cosh^2(t/\ell)$, is that they do not cover all dS^4. Since dS^{n+1} is a maximally symmetric spacetime we have

$$R_{\mu\nu\rho\alpha} R^{\mu\nu\rho\alpha} = \frac{2}{n(n+1)} R^2, \quad R = R = (n+1)n/\ell^2, \tag{1.46}$$

we conclude that the singularities for the parametrizations that do not cover the hole spacetime are actually coordinate singularities.

1.2.1 The action for dS^{n+1}

The Einstein-Hilbert action is defined as

$$S_{EH}[g_{\mu\nu}] = \frac{1}{16\pi G_N^{(n)}} \int \mathrm{d}^n x \sqrt{-\det g}\, (R - 2\Lambda). \tag{1.47}$$

Instead of dealing with the general variation problem, let us take a short cut and consider a metric on the form

$$\mathrm{d}s^2 = -N(t)\mathrm{d}t^2 + \mathfrak{a}^2(t) \mathrm{d}\Omega_3^2. \tag{1.48}$$

Here, the scale factor $\mathfrak{a}(t)$ has dimensions of length and the dimensionless function $N(t)$ is known as the lapse function. We have that

$$R = \frac{6}{N^2}\left(\frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} + \left(\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right)^2 - \frac{\dot{N}}{N}\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right) + \frac{6}{\mathfrak{a}^2}, \tag{1.49}$$

and

$$\mathrm{d}^4x\sqrt{-\det g} = \mathrm{d}t\mathrm{d}^3\theta\sqrt{\det\gamma}N\mathfrak{a}^3, \tag{1.50}$$

where γ_{ab} is the metric on S^3. The action becomes

$$S_{EH}[\mathfrak{a}, N] = \int \mathrm{d}t\left[L(\mathfrak{a}, \dot{\mathfrak{a}}, N) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2\lambda}\frac{\mathfrak{a}^2\dot{\mathfrak{a}}}{N}\right)\right], \tag{1.51}$$

with

$$L(\mathfrak{a}, \dot{\mathfrak{a}}, N) = \frac{1}{2\lambda}\left(-\frac{\mathfrak{a}\dot{\mathfrak{a}}^2}{N} + N\mathfrak{a} - \frac{\Lambda}{3}N\mathfrak{a}^3\right), \quad \lambda = \frac{8\pi G_N^{(3)}}{6\mathrm{Area}(S^3)} = \frac{2}{3\pi}G_N^{(3)}. \tag{1.52}$$

Now, the total derivative does not affect the Euler-Lagrange equations but modifies the variational problem. It is easy to see that the total derivative will give us a factor proportional to $\delta\dot{\mathfrak{a}}$. This is not suitable for a Dirichlet boundary condition. Therefore, the correct action must be

$$S[\mathfrak{a}, N] = S_{EH}[\mathfrak{a}, N] - \int \mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2\lambda}\frac{\mathfrak{a}^2\dot{\mathfrak{a}}}{N}\right). \tag{1.53}$$

The second term, in its general form, is known as the Gibbons-Hawking-York term[3].

Notice that the Lagrangian does not depend on $\dot{N}$, i.e. it is not dynamical, and thus $\frac{\partial L}{\partial \dot{N}} = 0$. The Euler-Lagrange reduces to $\frac{\partial L}{\partial N} = 0$ and gives

$$\frac{1}{N}\left(\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right)^2 = \frac{\Lambda}{3} - \frac{1}{\mathfrak{a}^2}. \tag{1.54}$$

The equation for the scale factor result

$$2\frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} + \left(\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right)^2 - 2\dot{N}\frac{\dot{\mathfrak{a}}}{\mathfrak{a}} = N^2\left(\Lambda - \frac{1}{\mathfrak{a}^2}\right). \tag{1.55}$$

After setting $N = 1$, we find the same 00, ij equations for $k = 1$. Therefore, we get the same expressions

$$\left(\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right)^2 = \frac{\Lambda}{3} - \frac{1}{\mathfrak{a}^2}, \quad \frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} = \frac{\Lambda}{3}, \tag{1.56}$$

[3] I refer to "A short note on the boundary term for the Hilbert action", Modern Physics Letters A 2014 29:08b y T. Padmanabhanand and "Robin Gravity" J. Phys.: Conf. Ser. 883 012011 by Krishnan, Maheshwari and Bala Subramanian for a detail discussion of this term.

but now we are dealing with a boundary problem with a constraint. For $\Lambda > 0$ and $\ell = \sqrt{3/\Lambda}$, consider the solution

$$\mathfrak{a}(t) = A\cosh(t/\ell) + B\sinh(t/\ell). \tag{1.57}$$

Notice that it can be written as

$$\mathfrak{a}(t) = \alpha e^{t/\ell} + \beta e^{-t/\ell}, \tag{1.58}$$

with

$$\alpha = \frac{A+B}{2}, \quad \beta = \frac{A-B}{2} \tag{1.59}$$

The constraint gives $4\alpha\beta = \ell^2$. Therefore

$$\mathfrak{a}(t) = \left(\alpha + \frac{\ell^2}{4\alpha}\right)\cosh(t/\ell) + \left(\alpha - \frac{\ell^2}{4\alpha}\right)\sinh(t/\ell). \tag{1.60}$$

Notice how the constraint changes the boundary problem! For the final state we consider $t_2 \gg \ell$ and $\mathfrak{a}_2 = \alpha_* e^{t_2/\ell}$. If we choose the initial state at $t = 0$, we have that

$$\mathfrak{a}_1 = \alpha_* + \frac{\ell^2}{4\alpha_*}. \tag{1.61}$$

We can invert to find

$$\alpha_* = \frac{\mathfrak{a}_1}{2} \pm \frac{1}{2}\sqrt{\mathfrak{a}_1^2 - \ell^2}, \tag{1.62}$$

with $\mathfrak{a}_1 \geq \ell$. Hence, after specifying the final state, the boundary problem traduces to an initial condition problem[4]. Notice that for $\mathfrak{a}_1 = \ell$ we obtain $\alpha_* = \ell/2$ and therefore we get dS^4 in global coordinates. On the other hand, consider the initial condition $\mathfrak{a}_1 \gg \ell$, then $\alpha_* \approx 0$ for the negative root or $\alpha_* \approx \mathfrak{a}_1$.

We end this discussion by analyzing the problem in phase space. The momenta are

$$\mathfrak{p} = \frac{\partial L}{\partial \dot{\mathfrak{a}}} = -\frac{1}{\lambda}\frac{\mathfrak{a}\dot{\mathfrak{a}}}{N}, \quad P = \frac{\partial L}{\partial \dot{N}} = 0. \tag{1.63}$$

The Hamiltonian result

$$H = N\mathcal{H}, \tag{1.64}$$

with

$$\mathcal{H}(\mathfrak{a}, \mathfrak{p}) = -\frac{\lambda}{2\mathfrak{a}}\mathfrak{p}^2 + \frac{\Lambda}{6\lambda}\mathfrak{a}^3 - \frac{1}{2\lambda}\mathfrak{a}. \tag{1.65}$$

Then, the action becomes

$$S[\mathfrak{a}, \mathfrak{p}, N] = \int dt\, (p\dot{\mathfrak{a}} - N\mathcal{H}(\mathfrak{a}, \mathfrak{p})). \tag{1.66}$$

Wee see that the equation for N gives $\mathcal{H} = 0$. This is known as the Hamiltonian constraint and gives (1.54). Hence, we interpret N as a Lagrange multiplier.

[4]If instead we do not think of the problem as a boundary one, we will arrive at the same conclusion. This is due to the fact that the system is the universe not part of it. Hence, the boundary problem mathematically accommodates the physical problem and shows how it reduces to an initial condition problem. Of course this is not just matter of elegance, quantum mechanics demand this framework.

1.2.2 The inflaton

Let us introduce a scalar field ϕ, the inflanton field, with potential $V(\phi)$. The goal is to replace the dynamics of the cosmological constant with this field. Then, to the gravitational action with $\Lambda = 0$ we add

$$S = S_g - \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,(\nabla_\mu\phi\nabla^\mu\phi + 2V(\phi)). \tag{1.67}$$

Since we are interested in a Dirichlet boundary problem, S_g has a Gibbons-Hawking-York term. We again consider the metric

$$\mathrm{d}s^2 = -N(t)\mathrm{d}t^2 + \mathfrak{a}^2(t)\mathrm{d}\Omega_3^2, \tag{1.68}$$

and $\phi = \phi(t)$ due the symmetries of space. Then, the action for the inflaton is

$$S[\phi] = \mathrm{Area}(S^3)\int \mathrm{d}t\,\left(\frac{\mathfrak{a}^3}{2N}\dot{\phi}^2 - N\mathfrak{a}^3V(\phi)\right). \tag{1.69}$$

Let $\phi(t) = \varphi(t)/\sqrt{\mathrm{Area}(S^3)}$ and $\mathcal{V}(\varphi) = \mathrm{Area}(S^3)V(\varphi/\sqrt{\mathrm{Area}(S^3)})$. Then, the total action gives

$$S[\mathfrak{a},\varphi] = \int \mathrm{d}t\left[-\frac{\mathfrak{a}\dot{\mathfrak{a}}^2}{2\lambda N} + \frac{N}{2\lambda}\mathfrak{a} + \frac{\mathfrak{a}^3}{2N}\dot{\varphi}^2 - N\mathfrak{a}^3\mathcal{V}(\varphi)\right]. \tag{1.70}$$

The equations are

$$\begin{aligned}
\frac{1}{N^2}\left(\left(\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right)^2 - \lambda\dot{\varphi}^2\right) &= 2\lambda\mathcal{V}(\varphi) - \frac{1}{\mathfrak{a}^2},\\
2\frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} + \left(\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right)^2 - 2\dot{N}\frac{\dot{\mathfrak{a}}}{\mathfrak{a}} &= N^2\left(6\lambda\mathcal{V}(\varphi) - \frac{1}{\mathfrak{a}^2}\right) - 3\lambda\dot{\varphi}^2,\\
\ddot{\varphi} + 3\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\dot{\varphi} - \dot{\varphi}\dot{N} &= -N^2\mathcal{V}'(\varphi).
\end{aligned} \tag{1.71}$$

For $N = 1$ we obtain

$$\begin{aligned}
\left(\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right)^2 &= \lambda\dot{\varphi}^2 + 2\lambda\mathcal{V}(\varphi) - \frac{1}{\mathfrak{a}^2},\\
2\frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} + \left(\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right)^2 &= 6\lambda\mathcal{V}(\varphi) - 3\lambda\dot{\varphi}^2 - \frac{1}{\mathfrak{a}^2},\\
\ddot{\varphi} + 3\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\dot{\varphi} + \mathcal{V}'(\varphi) &= 0.
\end{aligned} \tag{1.72}$$

We can rewrite the equations as

$$\begin{aligned}
\left(\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right)^2 &= \lambda\dot{\varphi}^2 + 2\lambda\mathcal{V}(\varphi) - \frac{1}{\mathfrak{a}^2},\\
\frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} &= 2\lambda(\mathcal{V}(\varphi) - \dot{\varphi}^2),\\
\ddot{\varphi} + 3\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\dot{\varphi} + \mathcal{V}'(\varphi) &= 0.
\end{aligned} \tag{1.73}$$

Notice that if

$$\mathcal{V}(\varphi) = \mathcal{V}_0, \tag{I.74}$$

a solution for the field is $\varphi = \varphi_* = const$. The remaining equations are

$$\begin{aligned} \left(\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right)^2 &= 2\lambda\mathcal{V}_0 - \frac{1}{\mathfrak{a}^2}, \\ \frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} &= 2\lambda\mathcal{V}_0. \end{aligned} \tag{I.75}$$

Thus we see that

$$\Lambda = 6\lambda\mathcal{V}_0. \tag{I.76}$$

For $\mathcal{V}_0 > 0$ we have de Sitter solution with

$$2\lambda\mathcal{V}_0 = \frac{1}{\ell^2} \tag{I.77}$$

Clearly for this case, the initial and final states of the inflaton are the same.

Chapter 2

Quantum aspects

2.1 Canonical quantization

Following Dirac's work on the quantization of constraint systems[1], the wave function of the system with scalar factor and scalar field is $\Psi(\mathfrak{a}, \varphi)$ and must satisfy

$$\hat{\mathcal{H}}\Psi(\mathfrak{a}, \varphi) = 0. \tag{2.1}$$

This requieres to quantize

$$\mathcal{H} = -\frac{\lambda}{2\mathfrak{a}}\mathfrak{p}^2 + \frac{1}{2a^3}p^2 + \mathfrak{a}^3\mathcal{V}(\varphi) - \frac{1}{2\lambda}\mathfrak{a}, \tag{2.2}$$

where p is the momentum conjugate to φ. Then, in the Schrödinger picture we have that

$$\mathfrak{p} \to -i\hbar\frac{\partial}{\partial\mathfrak{a}}, \quad p \to -i\hbar\frac{\partial}{\partial\varphi}. \tag{2.3}$$

Since

$$\frac{1}{\mathfrak{a}}\mathfrak{p}^2 = \mathfrak{p}\frac{1}{\mathfrak{a}}\mathfrak{p} = \mathfrak{p}^2\frac{1}{\mathfrak{a}}, \tag{2.4}$$

there is an ambiguity for the quantization. For this reason we consider

$$\frac{1}{\mathfrak{a}}\mathfrak{p}^2 \to -\frac{\hbar}{\mathfrak{a}^{s+1}}\partial_\mathfrak{a}(\mathfrak{a}^s\partial_\mathfrak{a}\cdot), \tag{2.5}$$

where s is a real parameter. The resulting equation

$$\frac{\hbar^2\lambda}{2\mathfrak{a}^{s+1}}\partial_\mathfrak{a}(\mathfrak{a}^s\partial_\mathfrak{a}\Psi) - \frac{\hbar^2}{2\mathfrak{a}^3}\partial_\varphi^2\Psi + \left(\mathfrak{a}^3\mathcal{V}(\varphi) - \frac{1}{2\lambda}\mathfrak{a}\right)\Psi = 0, \tag{2.6}$$

is the Wheeler-DeWitt (WdW) equation for the system. Let us search for solutions where the potential is approximately flat, i.e. approximately constant. Classically this

[1]The reference is P.A.M. Dirac: Lectures on Quantum Mechanics, Belfer Graduate School of Science Monographs, Vol. 2 (Yeshiva Univ., New York 1964)

corresponds to an approximately constant value of the scalar field. This approximation allow us to drop the term $\partial^2_\varphi \Psi$ and the equation reduces to

$$\frac{\hbar^2}{2\mathfrak{a}^{s+1}}\partial_\mathfrak{a}(\mathfrak{a}^s\partial_\mathfrak{a}\Psi) + \left(\frac{\mathcal{V}(\varphi)}{\lambda}\mathfrak{a}^3 - \frac{1}{2\lambda^2}\mathfrak{a}\right)\Psi = 0. \tag{2.7}$$

This can be interpreted as a one dimensional quantum system

$$-\frac{\hbar^2}{2\mathfrak{a}^{s+1}}\frac{\partial}{\partial\mathfrak{a}}\left(\mathfrak{a}^s\frac{\partial}{\partial\mathfrak{a}}\Psi\right) + U_{\text{eff}}\Psi = 0, \tag{2.8}$$

where

$$U_{\text{eff}} = \frac{1}{2\lambda^2}\mathfrak{a} - \frac{\mathcal{V}(\varphi)}{\lambda}\mathfrak{a}^3 \tag{2.9}$$

The potential for $\mathcal{V}(\varphi) > 0$ is plotted in figure 2.1. It vanishes for $\mathfrak{a} = 0$ and $\mathfrak{a}_* = 1/\sqrt{2\lambda\mathcal{V}(\varphi)}$ and its maximum is located at $\mathfrak{a}_{\text{max}} = \mathfrak{a}_*/\sqrt{3}$.

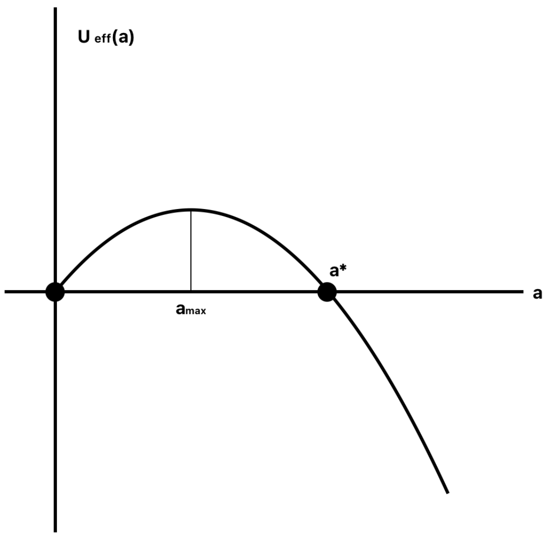

Figure 2.1: Effective potential plot.

If we follow the standard rules of ordinary quantum mechanics, we must demand that the wavefunction of the universe must be normalizable. This ensure a probabilistic interpretation of the wavefunction. Normalization in practice demands to impose a particular boundary condition. On the other hand, since physical observables are Hermitian, this restricts the choice of boundary condition as well. Apart from these mathematical requirements, it will the comparison with experiments that decide which boundary condition is adequate.

As an example, consider the time-independent Schrödinger equation of a particle with potential that has spherical symmetry. Then, we choose spherical coordinates and therefore the equation will have a coordinate singularity at $r = 0$. The equation is thus only valid for $r > 0$. Due to the rotational symmetry we use separation of variables. Then, the angular part of the wavefunction is given by spherical harmonics and we are left with the radial equation with an effective potential. The radial part, denoted as $R(r)$, is then written as $R(r) = u(r)/r$. Hence, the boundary conditions are considered for $u(r)$ instead of $R(r)$. This is valid since for n dimensions we will have

$$\int \mathrm{d}^n x \sqrt{g}\, |\psi|^2 = V(S^{n-1}) \int_0^\infty \mathrm{d}r\, r^{n-3} |u|^2. \tag{2.10}$$

For $n = 3$ we see that u must be normalizable. The problem reduces to find solutions of u that decay at $r \to \infty$ and are regular at $r = 0$. The substitution $R(r) = u(r)/r$ is the fruitful but consist more than just a re-writing. The radial equation contains a term of the form

$$-\frac{1}{r^{n-1}} \partial_r (r^{n-1} \partial_r R), \tag{2.11}$$

and after introducing u it becomes

$$-\frac{1}{r} \partial_r^2 u - (n-3) \left(\frac{1}{r^2} \partial_r u - \frac{1}{r^3} u \right). \tag{2.12}$$

For $n = 3$ it reduces the problem to a particle in half the real line. The kinetic part of the classical Hamiltonian of this particle has the form p^2/r and after the replacement $p \to -i\hbar\partial_r$, it must give $-\frac{1}{r}\partial_r^2$. Therefore, it chooses a particular ordering!

Notice the similarity of the resulting WdW equation given in equation (2.8). If this equation is derived from the classical Hamiltonian

$$\mathcal{H}_c = \frac{1}{\mathfrak{a}} \mathfrak{p}^2 + U_{\text{eff}}, \tag{2.13}$$

it is reasonable to consider a ordering with $s = 0$. Moreover, the analogy suggests that the wavefunction should be regular at $\mathfrak{a} = 0$.

We have to be careful since we are not dealing, say, with the toy model of the hydrogen atom, the system of interest is the universe[2]. Other physical requirements must be take into account. The quantum theory in a classical limit must reproduce a classical universe. Bear in mind that in the WdW there is no time and therefore it must emerge by some mechanism. On the other hand, we know that general relativity in a quantum field theory point of view is not renormalizable and thus we are just dealing with an effective theory. This just means that our current description of interactions between gravity and matter is valid for low energies or equivalent large distances.

[2] Later we will see that the WdW equation has a solution for $s = -1$ not $s = 0$.

Then, to reproduce classical results, we first seek for solutions in the semiclassical limit. In practice this amounts to consider the WKB approximation. Then, we forget about normalizing the wave function and consider the ansatz

$$\Psi = e^{\frac{i}{\hbar}\mathcal{S}}, \tag{2.14}$$

with

$$\mathcal{S} = \mathcal{S}_0 + \hbar\mathcal{S}_1 + \hbar^2\mathcal{S}_2 + \dots. \tag{2.15}$$

Then, at order $\hbar^0$ we obtain

$$\frac{1}{2\mathfrak{a}}\left(\frac{\partial}{\partial\mathfrak{a}}\mathcal{S}_0\right)^2 + U_{\text{eff}} = 0, \tag{2.16}$$

and at order $\hbar$

$$\frac{\partial}{\partial\mathfrak{a}}\mathcal{S}_1 = \frac{i}{2}\left(\frac{1}{\frac{\partial}{\partial\mathfrak{a}}\mathcal{S}_0}\frac{\partial^2}{\partial\mathfrak{a}^2}\mathcal{S}_0 + \frac{s}{\mathfrak{a}}\right). \tag{2.17}$$

Notice that at $\hbar^0$ order, the ordering problem is not relevant. Moreover, the expression given in equation (2.16) can be interpreted as the Hamilton-Jacobi equation of a classical particle with zero energy from which the equation (2.8) can be derived. Thus, the classical momentum corresponds to

$$\mathfrak{p} = \frac{\partial}{\partial\mathfrak{a}}\mathcal{S}_0. \tag{2.18}$$

The equation (2.16) can be written as

$$\frac{\partial}{\partial\mathfrak{a}}\mathcal{S}_0^{\pm} = \pm\frac{\mathfrak{a}}{\lambda}\sqrt{\left(\frac{\mathfrak{a}}{\mathfrak{a}_*}\right)^2 - 1}. \tag{2.19}$$

Following the particle interpretation, a real classical momentum is only obtained for $\mathfrak{a} > \mathfrak{a}_*$. Then, we can expect that in this region we should recover a classical universe[3] Since $\mathfrak{p} = -\mathfrak{a}\dot{\mathfrak{a}}/\lambda$, we have

$$\dot{\mathfrak{a}} = \mp\sqrt{\left(\frac{\mathfrak{a}}{\mathfrak{a}_*}\right)^2 - 1}. \tag{2.20}$$

For an expanding universe we have $\dot{\mathfrak{a}} > 0$ and this corresponds to $\mathcal{S}_0^-$. A collapsing universe correspond to $\mathcal{S}_0^+$. Which universes are we talking about? They are solutions of equation (2.20). The equation can be written as

$$\left(\frac{\mathfrak{a}}{\mathfrak{a}_*}\right)^2 - \dot{\mathfrak{a}}^2 = 1. \tag{2.21}$$

[3] If the reader thinks that this is extremely hand wavy, I recommend reading arXiv:0909.2566 and DECOHERENCE IN QUANTUM COSMOLOGY by J.J. Haliwell.

A general solution is of the form

$$\mathfrak{a} = \mathfrak{a}_* \cosh(\xi(t)), \quad \dot{\mathfrak{a}} = \pm \sinh(\xi(t)). \tag{2.22}$$

For a expanding universe we can set $\xi(t) = t/\mathfrak{a}_*$ to obtain $\mathfrak{a} = \mathfrak{a}_* \cosh(t/\mathfrak{a}_*)$ with $\mathfrak{a}_* = 1/\sqrt{2\lambda\mathcal{V}(\varphi)}$. In this approximation we can have $2\lambda\mathcal{V}(\varphi) = 1/\ell^2$ and therefore the solutions is dS^4 in global coordinates. In order to have $\dot{\mathfrak{a}} > 0$ we must restrict the solution to $0 < t < \infty$. For $\dot{\mathfrak{a}} < 0$, we can consider the same solution with $-\infty < t < 0$.

The solution of $\mathcal{S}_0^{\pm}$ is

$$\mathcal{S}_0^{\pm}(\mathfrak{a}) = C_0(\varphi) \pm \frac{\mathfrak{a}_*^2}{3\lambda}\left(\left(\frac{\mathfrak{a}}{\mathfrak{a}_*}\right)^2 - 1\right)^{3/2}, \tag{2.23}$$

and the solution for $\mathcal{S}_1$ is

$$\mathcal{S}_1^{\pm} = C_1(\varphi) + \frac{i}{2}\left[(s+1)\ln\left(\frac{\mathfrak{a}}{\mathfrak{a}_*}\right) + \frac{1}{2}\ln\left(\frac{\mathfrak{a}^2}{\mathfrak{a}_*^2} - 1\right)\right]. \tag{2.24}$$

Thus, the solutions are of the form

$$\Psi_{WKB} \sim \frac{1}{\left(\frac{\mathfrak{a}}{\mathfrak{a}_*}\right)^{\frac{s+1}{2}}\left(\frac{\mathfrak{a}^2}{\mathfrak{a}_*^2} - 1\right)^{\frac{1}{4}}}\left[A_1 \mathrm{e}^{+i\frac{\mathfrak{a}_*^2}{3\lambda\hbar}\left(\frac{\mathfrak{a}^2}{\mathfrak{a}_*^2}-1\right)^{3/2}} + A_2 \mathrm{e}^{-i\frac{\mathfrak{a}_*^2}{3\lambda\hbar}\left(\frac{\mathfrak{a}^2}{\mathfrak{a}_*^2}-1\right)^{3/2}}\right], \tag{2.25}$$

where A_1 and A_2 are constants. We see that for $\frac{\mathfrak{a}}{\mathfrak{a}_*} \gg 1$ the wavefunction oscillates and for $\frac{\mathfrak{a}}{\mathfrak{a}_*} \ll 1$ it decays/grows. For the former case, $\exp\left(+i\frac{\mathfrak{a}_*^2}{3\lambda\hbar}\left(\frac{\mathfrak{a}^2}{\mathfrak{a}_*^2} - 1\right)^{3/2}\right)$ corresponds to a contracting universe and $\exp\left(-i\frac{\mathfrak{a}_*^2}{3\lambda\hbar}\left(\frac{\mathfrak{a}^2}{\mathfrak{a}_*^2} - 1\right)^{3/2}\right)$ an expanding universe.

Notice that this expression suggests that we should consider $s = -1$ rather than $s = 0$ as one naively expect.

Now that we argued that we can actually find a classical spacetime, let us try to actually solve the WdW equation without the semiclassical approximation. We write the equation as

$$\frac{\partial^2}{\partial\mathfrak{a}^2}\Psi + \frac{s}{\mathfrak{a}}\frac{\partial}{\partial\mathfrak{a}}\Psi - \frac{9\pi^2}{4}\frac{\mathfrak{a}^2}{\ell_p^4}\left(1 - \frac{\mathfrak{a}^2}{\mathfrak{a}_*^2}\right)\Psi = 0, \tag{2.26}$$

where ℓ_p is the Planck length. Since the scale factor has units of length the new variable $\sigma = \mathfrak{a}^2$ has units of area. For $\Psi(\mathfrak{a}, \varphi) = \psi(\sigma, \varphi)$, the equation becomes

$$\psi'' + \frac{1+s}{2\sigma}\psi' + \frac{9\pi^2}{16\sigma_p^2}\left(\frac{\sigma}{\sigma_*} - 1\right)\psi = 0, \quad \sigma_p = \ell_p^2, \quad \sigma_* = \mathfrak{a}_*^2, \tag{2.27}$$

and $\psi' = \partial_\sigma \psi$. We can simplify the expression further by setting

$$\zeta = \frac{\sigma}{\sigma_*} - 1, \quad \psi(\sigma, \varphi) = f(\zeta). \tag{2.28}$$

Notice that $\zeta = \zeta(\mathfrak{a}, \varphi)$. We obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} f + \frac{1+s}{2(\zeta+1)} \frac{\mathrm{d}}{\mathrm{d}\zeta} + \kappa^2 \zeta f = 0, \quad \kappa = \frac{3\pi\sigma_*}{4\sigma_p}. \tag{2.29}$$

For $s = -1$ we equation becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} f = -\kappa^2 \zeta f, \tag{2.30}$$

and the solution is given in terms of Airy's functions

$$f_{s=-1}(\zeta) = c_1 \mathrm{Ai}\left(-\kappa^{2/3}\zeta\right) + c_2 \mathrm{Bi}\left(-\kappa^{2/3}\zeta\right). \tag{2.31}$$

Physically we expect that $\sigma \geq 0$ and therefore $\zeta \geq -1$. In figure 2.2 the functions are plotted for $\kappa = 1$ and $\zeta \geq -1$. The functions oscillate for $\zeta \gg 0$. The Ai function

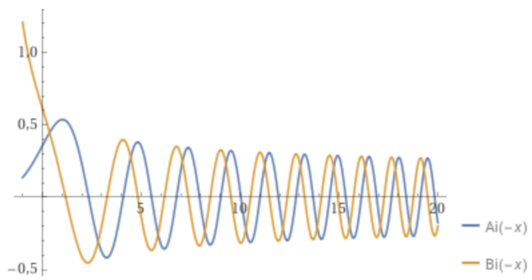

Figure 2.2: Plots of the solutions. In the graphic x stands for $-\kappa^{2/3}\zeta$ not $\mathfrak{a}$.

decays from $\zeta = 0$ toward $\zeta = -1$ and the Bi function grows from $\zeta = 0$ toward $\zeta = -1$. The region $-1 \leq \zeta \ll 0$ is the classical forbidden region since from

$$\zeta = \frac{\mathfrak{a}^2}{\mathfrak{a}_*^2} - 1, \tag{2.32}$$

we see that $0 \leq \mathfrak{a} \ll \mathfrak{a}_*$, compare with figure 2.1. Thus, for $0 \leq \mathfrak{a} \ll \mathfrak{a}_*$, the Ai function grows and the function Bi decays. We stress that the functions and their derivatives at $\zeta = -1$ does not vanish. However, the partial derivative with respect to the scale factor vanish! Indeed:

$$\frac{\partial}{\partial \mathfrak{a}} f_{s=-1}\bigg|_{\mathfrak{a}=0} = \frac{\partial \zeta}{\partial \mathfrak{a}}\bigg|_{\mathfrak{a}=0} \frac{\partial f_{s=-1}}{\partial \zeta}\bigg|_{\zeta=-1} = \frac{2\mathfrak{a}}{\mathfrak{a}_*^2}\bigg|_{\mathfrak{a}=0} \times const. = 0. \tag{2.33}$$

For $\zeta \gg 0$ and κ fixed, we have[4]

$$f_{s=-1}(\zeta) \sim \frac{1}{\sqrt{\pi}\kappa^{1/6}\zeta^{1/4}}\left[c_1 \cos\left(\frac{2}{3}\kappa\zeta^{3/2} - \frac{\pi}{4}\right) - c_2 \sin\left(\frac{2}{3}\kappa\zeta^{3/2} - \frac{\pi}{4}\right)\right] \tag{2.34}$$

The same result is obtained for $\zeta > 0$ fixed and $\kappa \gg 0$. This is the semiclassical limit since $\kappa \gg 0$ as $\ell_p \to 0$. Since $-\sin(x - \pi/4) = \cos(x + \pi/4)$, we can write

$$\begin{aligned} f_{s=-1}(\zeta) \quad &\sim \quad \frac{1}{\sqrt{\pi}\kappa^{1/6}\zeta^{1/4}} \\ &\times \left[(c_1 e^{-i\frac{\pi}{4}} + c_2 e^{i\frac{\pi}{4}})e^{+i\frac{2}{3}\kappa\zeta^{3/2}} + (c_1 e^{i\frac{\pi}{4}} + c_2 e^{-i\frac{\pi}{4}})e^{-i\frac{2}{3}\kappa\zeta^{3/2}}\right]. \end{aligned} \tag{2.35}$$

Due to the fact that

$$\frac{2}{3}\kappa\zeta^{3/2} = \frac{1}{\hbar}\frac{\mathfrak{a}_*^2}{3\lambda}\left(\frac{\mathfrak{a}^2}{\mathfrak{a}_*^2} - 1\right)^{3/2}, \tag{2.36}$$

we recover the WKB approximation given in equation (2.25).

Now we face the fundamental problem: how do we set c_1 and c_2?

As discussed for the radial wave function, to ensure normalization, we will search for regular solutions at $r = 0$ that decay at infinity. In our case, we find that both solutions are regular at $\zeta = -1$ and decay for $\zeta \to \infty$. The Airy functions are not square integrable in the range $\zeta \in [-1, \infty)$[5].

On the other hand, a general solution predict a superposition of expanding and contracting universes. In an ordinary quantum, it is standard to interpret that the wave function "collapse" due to a measurement. Of course we are assuming that the wavefunction is normalizable and thus the wavefunction corresponds to a probability amplitude. Moreover, the measurement is carried out by degrees of freedom outside the system. In our case, the wavefunction is not normalizable and external observers cannot (or don not) exist.

Regarding the non normalizability, we recall that in ordinary quantum mechanics these type of states are encounter in a scattering problem. We make sense of these states with the aid of the probability current and its conservation. In our case, we find from the WdW equation

$$\left(f\frac{\mathrm{d}f^*}{\mathrm{d}\zeta} - f^*\frac{\mathrm{d}f}{\mathrm{d}\zeta}\right) = const. \tag{2.37}$$

For the solution we get

$$-\kappa^{2/3}\left[(c_1 c_2^* - c_1^* c_2)\mathrm{Ai} \times \mathrm{Bi}' + (c_1^* c_2 - c_1 c_2^*)\mathrm{Ai}' \times \mathrm{Bi}\right] = const. \tag{2.38}$$

[4]For the expansions of the Airy functions see "Special functions and their applications" by Lebedev.
[5]Notice that only $\mathrm{Ai}(z)$ is square integrable in the range $z \in [0, \infty)$, see e.g. Airy Functions and Applications to Physics by Olivier Vallée and Manuel Soares.

The solution is found for a zero constant and $c_1 c_2^* = c_1^* c_2$. The result imply $c_1^2|c_2|^2 = |c_2|^2 c_1^2$ and thus we can only consider the constants to be real. Hence, the wavefunction must be real.

Recall that for a scattering problem one usually consider scattering states , linear combinations of time independent plane waves at given energy, and thus they are not normalizable. However, the probability current gives a method to compute the relation between the coefficients and find physical answers. Consider the problem of a particle beam arriving from $x \to -\infty$ hitting a finite barrier located near $x = 0$. The incoming wavefunction is a superposition of the source and reflect waves, i.e. $c\mathrm{e}^{ik_0x} + r(k_0)\mathrm{e}^{-ik_0x}$ and the outgoing or transmitted wave $t(k_0)\mathrm{e}^{ik_0x}$. The amplitudes $r(k_0)$ and $t(k_0)$ are the reflexion and transmission coefficients respectively. The probability current and its conservation gives $|c|^2 = |r(k_0)|^2 + |t(k_0)|^2$. Hence, we can write

$$P_r = \frac{|r(k_0)|^2}{|c|^2}, \quad P_t = \frac{|t(k_0)|^2}{|c|^2}, \quad 1 = P_r + P_t. \tag{2.39}$$

Interpreting P_r and P_t as conditional probabilities[6] of reflexion and transmission, we see that scattering states can give us physical information. The reason why these states can do that is because a normalizable wavefunction can be constructed as a wave packet of scattering sates. For $x \to -\infty$ and $x \to +\infty$, we will have

$$\psi_s + \psi_r = \int \mathrm{d}k\, A(k)(c(k)\mathrm{e}^{ikx} + r(k)\mathrm{e}^{-ikx}), \quad \psi_t = \int \mathrm{d}k\, A(k)t(k)\mathrm{e}^{ikx}. \tag{2.40}$$

Then, if $A(k)$ sharply peak at $k = k_0$,we obtain the scattering states.

Motivated by the above, let us consider the two simplest cases for c_1 and c_2 real: i) $c_2 = 0$ and ii) $c_1 = 0$. Thus we consider

$$\Psi_A = \frac{\mathrm{Ai}(-\kappa^{3/2}\zeta)}{\mathrm{Ai}(\kappa^{3/2})}, \quad \Psi_B = \frac{\mathrm{Bi}(-\kappa^{3/2}\zeta)}{\mathrm{Bi}(\kappa^{3/2})}. \tag{2.41}$$

At best, $|\Psi|^2$ can be interpreted as a conditional probability (density.) Let us consider the event of the creation of the universe at $\zeta = -1$ and interpret its probability to be $|\Psi(\zeta = -1)|^2$. Then, the conditional probability that it has its classical form $(\zeta \gg 0)$ given that was created at $\zeta = -1$ is $|\Psi|^2$. Since the universe has its classical form <u>because</u> it was created at $\zeta = -1$, the conditional probability can be naturally be interpreted as the probability of the universe being created.

The behavior of both wave functions[7] are summarized in table 2.1.

Hence, the conditional probabilities are

$$|\Psi_A|^2 \sim \mathrm{e}^{+\frac{4}{3}\kappa}, \quad |\Psi_B|^2 \sim \mathrm{e}^{-\frac{4}{3}\kappa}. \tag{2.42}$$

[6] See appendix A for the justification of this interpretation.

[7] For $\zeta \ll 0$ and κ fixed, we have $f_{s=-1}(\zeta) \sim \frac{1}{\sqrt{\pi}\kappa^{1/6}(-\zeta)^{1/4}}\left[\frac{c_1}{2}\mathrm{e}^{-\frac{2}{3}\kappa(-\zeta)^{3/2}} + c_2\mathrm{e}^{+\frac{2}{3}\kappa(-\zeta)^{3/2}}\right]$.

	$\zeta \ll 0$	$\zeta \gg 0$
Ψ_A	$\sim \frac{2}{(-\zeta)^{1/4}} e^{+\frac{2}{3}\kappa} e^{-\frac{2}{3}\kappa(-\zeta)^{3/2}}$	$\sim \frac{2}{\zeta^{1/4}} e^{+\frac{2}{3}\kappa} \cos\left(\frac{2}{3}\kappa\zeta^{3/2} - \frac{\pi}{4}\right)$
Ψ_B	$\sim \frac{1}{(-\zeta)^{1/4}} e^{-\frac{2}{3}\kappa} e^{+\frac{2}{3}\kappa(-\zeta)^{3/2}}$	$\sim -\frac{1}{\zeta^{1/4}} e^{-\frac{2}{3}\kappa} \sin\left(\frac{2}{3}\kappa\zeta^{3/2} - \frac{\pi}{4}\right)$

Table 2.1: Behavior of Ψ_A and Ψ_B

Working in a system of units in which $\lambda = 1$ and $\hbar = 1$, we have

$$\kappa = \frac{1}{2\mathcal{V}(\varphi)}, \tag{2.43}$$

and thus

$$|\Psi_A|^2 \sim e^{+\frac{2}{3}\frac{1}{\mathcal{V}(\varphi)}}, \quad |\Psi_B|^2 \sim e^{-\frac{2}{3}\frac{1}{\mathcal{V}(\varphi)}}. \tag{2.44}$$

On the other hand if we work in units i which $\ell_p = 1$ and since we obtain de Sitter, we obtain

$$\kappa = \frac{9\pi}{4}\frac{1}{\Lambda}. \tag{2.45}$$

Then

$$\frac{4}{3}\kappa = \frac{1}{4}\frac{12\pi}{\Lambda} = \pi\ell^2. \tag{2.46}$$

Thus, we can also write

$$|\Psi_A|^2 \sim e^{+\pi\ell^2}, \quad |\Psi_B|^2 \sim e^{-\pi\ell^2}. \tag{2.47}$$

The conditional probability for Ψ_A is enhanced for large ℓ (small but constant $\mathcal{V}(\varphi)$ or small Λ) and small ℓ (large but constant $\mathcal{V}(\varphi)$ or large Λ) for Ψ_B. Consider a potential given in figure 2.3. The regions around the points O, O', Q, Q' are approximately flat, i.e., constant. Then, φ is approximately constant in each region. Then, we see that $|\Psi_A|^2$ is enhanced for the points Q, Q' compared with O, O'. Moreover, among the pair Q, Q', the best option is Q. For $|\Psi_B|^2$, the points O, O' are preferred compared with Q, Q' and among the pair O, O', the best option is O'.

Since the physical scenario is the creation of a universe, we should expect that the potential should be at the lowest value possible. This is of course is a purely theoretical prejudice. Following this reasoning, we then should discard the solution Ψ_B since for Q the conditional probability is small. Hence, we arrive to the conclusion that Ψ_A is the best solution and we expect ℓ to be large.

Let us study Ψ_A in more detail.

The metric of the expanding universe can be written as

$$ds^2 = -dt^2 + \frac{\ell^2}{4}(e^{2t/\ell} + 2 + e^{-2t/\ell})\frac{1}{\left(1 + \frac{\rho^2}{4\ell^2}\right)^2}(d\rho^2 + \rho^2 d\Omega_2^2), \quad t > 0. \tag{2.48}$$

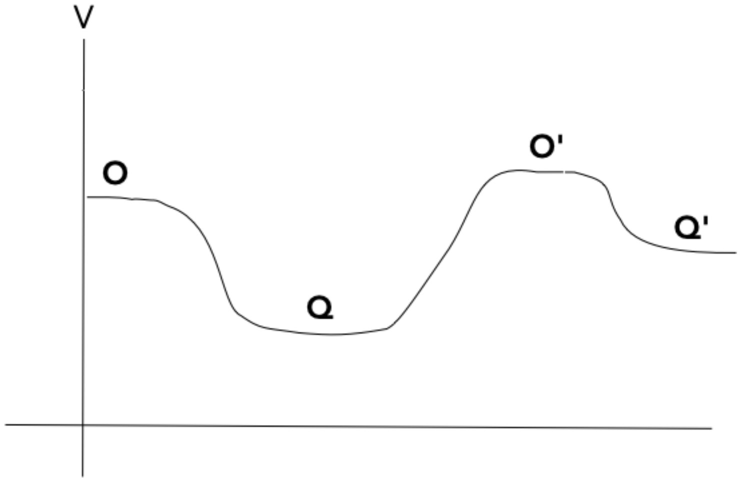

Figure 2.3: Example of a potential for the inflaton.

Let $\rho = 2\bar{\rho}$ and consider $\ell \to \infty$. Then

$$\mathrm{d}s^2 \approx -\mathrm{d}t^2 + \ell^2 \mathrm{e}^{2t/\ell} \mathrm{d}\mathbf{x}^2 \quad t > 0. \tag{2.49}$$

The spatial geometry becomes flat. For the contracting universe we just consider $-\infty < t < 0$. It remains to discuss which universe (expanding or contracting) is selected. The problem is basically how to overcome the fact that we cannot assume the "collapse" of the wavefunction.

The simplest possible solution is to pick one possibility and define that solution as the wavefunction of that universe. This seems ad hoc but notice that the semiclassical approximation tells us that the two universe do not interfear with each other.

From now on, we will focus on the expanding universe.

2.1.1 Path integral representation of Ψ_A

Let us now deal with a different situation. Consider the euclidean gravitational action

$$I_E[g_{\mu\nu}] = -\frac{1}{16\pi G_N} \int \mathrm{d}^4 x \sqrt{g} (R - 2\Lambda), \tag{2.50}$$

for compact manifolds, i.e. manifolds **without boundary**. The equations of motion are $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$. We search for solutions that satisfy $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with $\Lambda > 0$. Then, the equation reduces to $R = 4\Lambda$. Consider

$$\mathrm{d}s^2 = \ell^2 \mathrm{d}\Omega_4^2. \tag{2.51}$$

Then $R = 12/\ell^2$ and thus $\Lambda = 3/\ell^2$. Therefore, S^4_ℓ is a solution. The on-shell action result

$$I_E^* = -\frac{\Lambda \ell^4}{8\pi G_N}\text{Area}(S^4) = -\frac{3}{8\pi G_N}\ell^2 \frac{2\pi^{5/2}}{\Gamma(5/2)} = -\frac{\pi \ell^2}{G}. \tag{2.52}$$

The key point to notice is that

$$\left. \mathrm{e}^{-\frac{I_E^*}{\hbar}} \right|_{\ell_p = 1} = \mathrm{e}^{+\pi\ell^2} = |\Psi_A|^2. \tag{2.53}$$

This is very interesting since $\mathrm{e}^{-\frac{I_E^*}{\hbar}}$ corresponds to the semiclassical limit of the path integral

$$\int \mathcal{D}g_{\mu\nu}\, \mathrm{e}^{-\frac{I_E[g_{\mu\nu}]}{\hbar}}. \tag{2.54}$$

This suggests that Ψ_A may have also have an euclidean path integral representation. This is strange since gravity deals with Lorentzian manifolds. In order to shed some light to this possibility, consider the metric of S^4_ℓ

$$\mathrm{d}s^2 = \ell^2(\mathrm{d}\psi^2 + \sin^2\psi \mathrm{d}\Omega_3^2), \quad 0 \le \psi \le \pi. \tag{2.55}$$

The equator is given for $\psi = \pi/2$. Let $\psi = \bar{\psi} + \pi/2$, then

$$\mathrm{d}s^2 = \ell^2(\mathrm{d}\bar{\psi}^2 + \cos^2\bar{\psi}\mathrm{d}\Omega_3^2), \quad -\frac{\pi}{2} \le \bar{\psi} \le \frac{\pi}{2}. \tag{2.56}$$

Now let $\vartheta = \ell\bar{\psi}$ to obtain

$$\mathrm{d}s^2 = \mathrm{d}\vartheta^2 + \ell^2\cos^2(\vartheta/\ell)\mathrm{d}\Omega_3^2, \quad -\frac{\pi}{2}\ell \le \vartheta \le \frac{\pi}{2}\ell. \tag{2.57}$$

If we consider the analytical continuation $\vartheta = \pm it$ we get

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \ell^2\cosh^2(t/\ell)\mathrm{d}\Omega_3^2, \tag{2.58}$$

which is the metric of dS^4. Hence, we conclude that the metric of the euclidean gravitational action S_E associated to Ψ_A[8] must be in general complex in order to reproduce the classical universe. To be more precise, by a complex metric, we mean that the time coordinate must be complex with a scale factor always real. Let z be the complex time. Then, the line segment is of the form

$$\mathrm{d}s^2 = \mathrm{d}z^2 + \mathfrak{a}^2(z)\mathrm{d}\Omega_3^2, \quad (\mathfrak{a}(z))^* = \mathfrak{a}(z). \tag{2.59}$$

Now, let us consider a Dirichlet boundary problem for the complex metric given above. Then, the problem reduces to a Dirichlet boundary problem for $\mathfrak{a}(z)$. The action is considerer to be the Einstein-Hilbert action plus the Gibbons-Hawking-York term. To obtain this action, first consider the Lorentzian version

$$S_L = \frac{1}{2\lambda}\int \mathrm{d}tN\left(-\frac{\mathfrak{a}\dot{\mathfrak{a}}^2}{N^2} + \mathfrak{a} - \frac{\Lambda}{3}\mathfrak{a}^3\right). \tag{2.60}$$

[8] Recall that the action I_E is related to $|\Psi_A|^2$.

Then, we consider $N\mathrm{d}t = -i\mathcal{N}\mathrm{d}z$ to obtain

$$S_L = +i\int \mathrm{d}z\left[-\frac{\mathfrak{a}}{\mathcal{N}}\left(\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}z}\right)^2 - \mathcal{N}\mathfrak{a} + \mathcal{N}\frac{\Lambda}{3}\mathfrak{a}^3\right]. \tag{2.61}$$

Then, the euclidean action corresponds to

$$S_E = \frac{1}{2\lambda}\int \mathrm{d}z\left[-\frac{\mathfrak{a}}{\mathcal{N}}\left(\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}z}\right)^2 - \mathcal{N}\mathfrak{a} + \frac{\Lambda}{3}\mathcal{N}\mathfrak{a}^3\right]. \tag{2.62}$$

We see that $\mathrm{e}^{\frac{i}{\hbar}S_L} = \mathrm{e}^{-\frac{1}{\hbar}S_E}$ and the line segment is of the form

$$\mathrm{d}s^2 = \mathcal{N}(z)\mathrm{d}z^2 + \mathfrak{a}^2(z)\mathrm{d}\Omega_3^2. \tag{2.63}$$

The constraint and the equation result

$$\left(\frac{1}{\mathcal{N}}\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}z}\right)^2 = 1 - \frac{\Lambda}{3}\mathfrak{a}^2, \quad \frac{1}{\mathfrak{a}}\frac{\mathrm{d}^2\mathfrak{a}}{\mathrm{d}z^2} - \frac{1}{\mathfrak{a}}\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}z}\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}z} = -\frac{\Lambda}{3}\mathcal{N}^2. \tag{2.64}$$

For $\mathcal{N} = 1$ result

$$\left(\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}z}\right)^2 = 1 - \frac{\Lambda}{3}\mathfrak{a}^2, \quad \frac{1}{\mathfrak{a}}\frac{\mathrm{d}^2\mathfrak{a}}{\mathrm{d}z^2} - = -\frac{\Lambda}{3}. \tag{2.65}$$

For $\Lambda = 3/\ell^2$, we see that

$$\mathfrak{a} = \ell\sin(z/\ell), \tag{2.66}$$

is a complex solution. Then, the on-shell action result

$$S_E^* = \frac{1}{\lambda}\int_\gamma \mathrm{d}z\left(-\mathfrak{a} + \frac{\mathfrak{a}^3}{\ell^2}\right). \tag{2.67}$$

Consider the path γ to be

$$\gamma = \begin{cases} z = \tau & 0 \le \tau \le \frac{\pi}{2}\ell \\ z = \frac{\pi}{2}\ell + it & 0 \le t \le t_f \end{cases}. \tag{2.68}$$

Then we see that

$$\mathfrak{a} = \begin{cases} \ell\sin(\tau/\ell) & 0 \le \mathfrak{a} \le \ell \\ \ell\cosh(t/\ell) & \ell \le \mathfrak{a} \le \ell\cosh(t_f/\ell) \end{cases}. \tag{2.69}$$

In geometric terms, this corresponds to glue half of S_ℓ^4 with a region $(0 \le t \le t_f)$ of dS^4. This is depicted in figure 2.4. Recall that the constraints change the boundary problem to an initial condition problem. The initial condition now traduces in the euclidean region where

$$\mathfrak{a}(\tau = 0) = 0, \quad \frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\tau}(\tau = 0) = +1. \tag{2.70}$$

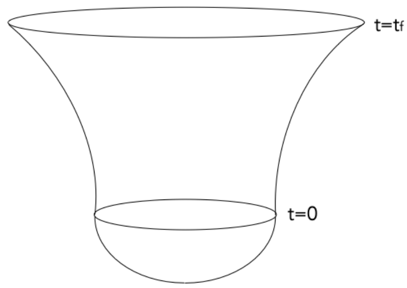

Figure 2.4: Gluing of half S^4_ℓ with the upper half of dS^4.

Then, the final state is located in the Lorentzian region where $t = t_f$, i.e. the path ends at t_f and thus there is a boundary. Evaluating the on-shell action, the real part gives

$$-\Re\left(\frac{S_E^*}{\hbar}\right) = +\frac{2}{3}\hbar\kappa, \tag{2.71}$$

and the imaginary part is of the form

$$-\Im\left(\frac{S_E^*}{\hbar}\right) = -\frac{\ell^2}{\lambda\hbar}\int_0^{u_f} \mathrm{d}u\,(\cosh^3 u - \cosh u), \tag{2.72}$$

where $u = t/\ell$. The integral gives

$$\begin{aligned}
\int_0^{u_f} \mathrm{d}u\,(\cosh^3 u - \cosh u) &= \int_0^{u_f} \mathrm{d}u\,\cosh u(\cosh^2 u - 1),\\
&= \int_0^{u_f} \mathrm{d}(\sinh u)\,\sinh^2 u,\\
&= \frac{1}{3}\sinh^3 u_f,\\
&= \frac{\sinh u_f}{3}(\cosh^2 u_f - 1),\\
&= \pm\frac{1}{3}(\cosh^2 u_f - 1)^{3/2},\\
&= \pm\frac{1}{3}\left(\frac{\mathfrak{a}_f^2}{\mathfrak{a}_*^2} - 1\right)^{3/2},\\
&= \pm\frac{1}{3}\zeta_f^{3/2}.
\end{aligned} \tag{2.73}$$

Therefore

$$-\Im\left(\frac{S_E^*}{\hbar}\right) = \mp\frac{2}{3}\kappa\zeta_f^{3/2}. \tag{2.74}$$

Hence,

$$\frac{S_{E,\pm}^*}{\hbar} = +\frac{2}{3}\hbar\kappa \mp i\frac{2}{3}\kappa\zeta_f^{3/2}. \tag{2.75}$$

In order to reproduce the WKB approximation for an expanding universe we consider only the $+$ result. We conclude that

$$\Psi_A \approx \mathrm{e}^{-\frac{S_{E,+}^*}{\hbar}} = \mathrm{e}^{+\frac{2}{3}\kappa - i\frac{2}{3}\kappa\zeta_f^{3/2}}. \tag{2.76}$$

Hence we see that the wavefunction can be interpreted as

$$\Psi_A \propto \lim_{\hbar\to 0}\int \mathcal{D}g_{\mu\nu}\,\mathrm{e}^{-\frac{1}{\hbar}S_E} \sim \sum \mathrm{e}^{-\frac{1}{\hbar}S_E^*}. \tag{2.77}$$

where the sum is over all the possible euclidean solutions that extremize the action, i.e. the sum over the saddles. This is analog to the procedure of finding the ground sate of an ordinary quantum mechanical problem via an euclidean path integral. In appendix B, this is discussed in detail for the harmonic oscillator. This simple example show us that the euclidean path integral is a mathematical prescription to define the ground state.

Hence, for a general prescription, the sum of saddles is over complex geometries (complex in time) that in the euclidean region are compact and that in the real time region reproduces a universe at some particular time, i.e. t_f. With this picture, we see that the problem is a one-boundary problem of spatial real metrics with the boundary defined in the Lorentzian sector at t_f. The one boundary problem is actually refer as the no-boundary proposal by Hartley and Hawking[9]. The discussed complex geometry is the prototype for all no-boundary solutions. It corresponds to a specific saddle.

On the other hand, since a boundary problems are changed if constraints are involved, our problem can be interpreted as an initial condition problem. If we set the boundary, then the task is to find the complex geometries that will give us the final state. Of course, we seek for a particular class of geometries of the form given in equation (2.63). In the euclidean region the complex geometry must satisfy $\mathfrak{a}(z=0)=0$. Then, the constraint for general $\mathcal{N}$ indicates that for $\mathfrak{a}(z=0)=0$, we must have $\frac{1}{\mathcal{N}}\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}z}(z=0)=\pm 1$. Thus, from the results of the prototype, we must consider $\frac{1}{\mathcal{N}}\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}z}(z=0)=+1$. Hence the path integral is of the form

$$\int \mathrm{d}N \int \mathcal{D}\mathfrak{a}\,\mathrm{e}^{-\frac{1}{\hbar}S_E[N,\mathfrak{a}]}. \tag{2.78}$$

The path for the ordinary integral for $\mathcal{N}$ must be determined.

[9] See Hartle, J.; Hawking, S. (1983)."Wave function of the Universe". Physical Review D. 28 (12): 2960. Don Page called the no-boundary proposal a one boundary proposal in arXiv:hep-th/0610121.

So far, the path integral representation of Ψ_A has been build with only gravity. If we introduced the inflaton field, which its goal is give a dynamical nature of the cosmological constant, the initial step is to consider the euclidean action

$$S_E = \int \mathrm{d}z \left[-\frac{\mathfrak{a}}{2\lambda\mathcal{N}} \left(\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}z} \right)^2 - \frac{\mathcal{N}}{2\lambda}\mathfrak{a} + \frac{\mathfrak{a}^3}{2\mathcal{N}} \left(\frac{\mathrm{d}\varphi}{\mathrm{d}z} \right)^2 + \mathcal{N}\mathfrak{a}^3 \mathcal{V}(\varphi) \right], \quad (2.79)$$

where now φ is in general a complex field. The equation of motion for this field is

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}z^2} + 3\frac{1}{\mathfrak{a}}\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}z}\frac{\mathrm{d}\varphi}{\mathrm{d}z} - V' = 0. \quad (2.80)$$

Again, we consider solutions for a nearly flat potential. Then, the equation at $z = 0$, reduces to

$$\mathfrak{a}(z=0)\frac{\mathrm{d}^2\varphi}{\mathrm{d}z^2}(z=0) + 3\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}z}(z=0)\frac{\mathrm{d}\varphi}{\mathrm{d}z}(z=0) \approx 0, \quad (2.81)$$

which gives

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z}(z=0) \approx 0. \quad (2.82)$$

Then, we see that $\varphi(z = 0)$ is a free parameter, i.e. an approximately a complex constant.

The wavefunction that includes the inflaton with the complex geometry depicted in figure 2.4 is called the Hartley-Hawking wavefunction Ψ_{HH}.

The euclidean path integral representation proposal has faces several problems. Choosing a particular saddle with specific initial conditions may seem ad hoc. It is! But is a theoretical proposal, so its predictions must contrasted with experiments (if possible). The issues that we are now going to concentrate are purely theoretical: general relativity is not renormalizable and the euclidean path integral is not bounded from below. These issues (if any) are encoded in the dynamical part of the proposal, i.e. the action.

In real time, the Lagrangian of the gravitational action has been taken to be $\mathcal{L} = \frac{1}{16\pi G_N}(R-2\Lambda)$ (let us ignore the inflaton). If we want to want to quantize the theory in a quantum field theory approach, we split the metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + \sqrt{32\pi G_N} h_{\mu\nu}$. Then, for small G_N, $h_{\mu\nu}$ can be interpreted as the classical fluctuations around the background $\bar{g}_{\mu\nu}$. Let us consider $\Lambda = 0$, then the background is flat and the resulting action corresponds an infinite (two) derivative expansion of $h_{\mu\nu}$ and the terms (schematically) are proportional to $(\sqrt{G_N})^n / G_N h^{n-1}(\partial h)^2$ where $n = 2, 3, \ldots$, i.e.

$$S \sim \int \mathrm{d}^4x \sum_{n=2}^{\infty} G_N^{\frac{n-2}{2}} h^{n-1}(\partial h)^2 = \int \mathrm{d}^4x \left((\partial h)^2 + \sqrt{G_N} h^2(\partial h) + G_N h^3 (\partial h)^2 + \ldots\right). \quad (2.83)$$

At quadratic order in $h_{\mu\nu}$, i.e. $n = 2$, the action does not depend on G_N. Thus including the infinite terms, we see that the ones for $n > 2$ are suppressed. The field $h_{\mu\nu}$ can be thought as the generalization of the spin 1 one massless gauge field A_μ of electrodynamics and it is referred as to the graviton (massless with spin 2). The gauge transformation for A_μ: $A_\mu \to A_\mu + \partial_\mu \theta$ corresponds to an infinitesimal diffeomorphism transformation for $h_{\mu\nu}$: $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. For $n = 2$ the classical vacuum equations of motion (in some gauge) are $-\Delta h_{\mu\nu} = 0$ and thus we have wave solutions. These solutions are the gravitational waves.

If we now consider a non-vanish cosmological constant, we will obtain the same result but now the action has another contribution of infinite terms proportional the possible contractions of $h_{\mu\nu}$ with itself and the background metric. In this second expansion, one obtains a term of the form $\Lambda(h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} g^{\rho\sigma} h_{\rho\sigma})h^{\mu\nu}$. Naively we may interpreted it as a mass term but it cannot be since it would break diffeomorphism invariance.

Let us set $\Lambda = 0$ again. By writing $h_{\mu\nu}(x) = e_{\mu\nu}(k)\mathrm{e}^{-ik\cdot x}$ we find that each interaction vertex has a factor of $G_N^{\frac{n-2}{2}} k^2$ (of course we only consider $n > 2$). If we want to compute quantum correction to the graviton amplitudes we will face divergences for large momenta. This is not problematic. The standard model of particles phases the same issue. The difference is that by the addition of a finite set of countert-terms in the action, these UV divergences are removed. For the case of the graviton, we will need infinite set of counterterms. So we say that gravity, by these we mean the theory of the graviton, is not normalizable. This is not a monumental drawback since it makes physical predictions at low energies, i.e. large distances compared to the Planck length. Thus, we consider it to be an effective field theory. This is the main point. Then, in this point of view the Lagrangian $\frac{1}{16\pi G_N}(R - 2\Lambda)$ should consider also all possible Riemannian invariants such as $G_N R^2$ and so on[10].

For the no-boundary proposal, notice that we recover the classical spacetime via the semiclassical approximation. Mathematically, this corresponds to the saddle point approximation of the euclidean path integral. Since

$$\exp\left(-\frac{1}{\hbar}S_E\right) = \exp\left(-\frac{1}{16\pi G_N \hbar}\int \mathrm{d}^4 x \sqrt{g}(\ldots)\right), \tag{2.84}$$

we see that the weak field limit $G_N \to 0$ is compatible with the limit $\hbar \to 0$. Hence, Ψ_{HH} contemplates only the leading contribution of the weak field limit. This is not a problem. Since ℓ is the characteristic length of the system and $\ell_p = \sqrt{G_N \hbar}$, then in the effective field approach this means that the description should be valid for $\ell \gg \ell_p$. This is indeed true. We have argued that, via the conditional probability, ℓ should be large.

[10] See Introduction to the Effective Field Theory Description of Gravity arXiv:gr-qc/9512024 by John F. Donoghue and also Effective Field Theory, Past and Future arXiv:0908.1964 by Steven Weinberg.

Regarding the conformal problem, consider the Weyl transformation $\tilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x)$. Then, we obtain that

$$\sqrt{\det \tilde{g}}R(\tilde{g}) = \sqrt{g}[\Omega^2 R(g) + 6\nabla_\mu\Omega\nabla^\mu\Omega - 6\nabla_\mu(\Omega\nabla^\mu\Omega)]. \tag{2.85}$$

So that the euclidean action, schematically is of the form

$$S_E[\tilde{g}] \sim S_E[g] - \int \sqrt{g}\nabla_\mu\Omega\nabla^\mu\Omega. \tag{2.86}$$

This means that $e^{-\frac{1}{\hbar}S_E[\tilde{g}]}$ can grow (rather than decay) for large values of Ω. The way to get around this problem is to recall that the no-boundary prescription is a prescription to define a path integral representation of the wavefunction. Moreover, it is used only in the semiclassical approximation and therefore we focus only on the saddles. It does not have the goal the make euclidean gravity theory a well defined theory[II].

Now we are in position to discuss the phenomenological triumphs and issues of Ψ_{HH}.

[II]Nevertheless some proposals to solve this problem has been suggested in: "Path integrals and the indefiniteness of the gravitational action", Nuclear Physics B138 (1978) 141-150 by Gibbons, Hawking and Perry, "The path integral measure, conformal factor problem and stability of the ground state of quantum gravity", Nuclear Physics B341 (1990) 187—212 by Mazur and Mottola and more recently in "The Canonical Ensemble Reloaded: The Complex-Stability of Euclidean quantum gravity for Black Holes in a Box", J. High Energ. Phys. 2022, 215, arXiv:2202.11786 by Marolf and Santos.

Chapter 3

Phenomenology

Let us start from scratch and consider the equation (no Λ)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}. \tag{3.1}$$

If we apply this equation to the universe, all that can be inside of it is modeled as a fluid. Like air molecules in a vessel. Notice that planets are basically a point in this cosmological fluid. Consider now

$$\nabla^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 8\pi G_N \nabla^\mu T_{\mu\nu}. \tag{3.2}$$

By construction (due to metric compatibility) the right hand side is equal to zero and thus we see that the energy-momentum tensor is conserved, i.e. $\nabla^\mu T_{\mu\nu} = 0$. Notice that equation can be written as

$$R_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \quad T = g^{\mu\nu} T_{\mu\nu}. \tag{3.3}$$

The metric ansatz is[1]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t) \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega^2 \right), \tag{3.4}$$

and the energy-momentum tensor is consider to be a perfect fluid. Then we have

$$T_{00} = \rho, \quad T_{ij} = g_{ij} p, \quad T_{0i} = 0, \tag{3.5}$$

where ρ is the energy density and p the pressure. The equation of state is assume to be $p = w\rho$. The trace and the conservation of $T_{\mu\nu}$ gives

$$T = -\rho + 3p, \quad \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}. \tag{3.6}$$

[1] By the way this metric is referred as to the Friedmann-Robertson-Walker (FRW) metric.

Then, equations to solve are

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G_N}{3}\rho - \frac{k}{a^2}, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G_N}{3}(\rho + 3p), \\ \frac{\dot{\rho}}{\rho} &= -3(1+w)\frac{\dot{a}}{a}. \end{aligned} \tag{3.7}$$

Recall that the first equation is the constraint. In order to solve the third equation, let t_0 the time of today and thus the solution is

$$\rho(t) = \rho_0 \left(\frac{a(t)}{a_0}\right)^{-3(1+w)}, \tag{3.8}$$

where $\rho_0 = \rho(t_0)$ and $a_0 = a(t_0)$. Cosmologist use the Hubble parameter defined as

$$H(t) = \frac{\dot{a}}{a}, \tag{3.9}$$

and thus $H_0 = H(t_0)$ is its value today. The remaining equations become

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2}, \quad \dot{H} = -4\pi G_N(1+w)\rho + \frac{k}{a^2}. \tag{3.10}$$

For a realistic model of the universe, we should consider a collection of perfect fluids and thus the equations to solve are

$$H^2 = \frac{8\pi G_N}{3}\varrho - \frac{k}{a^2}, \quad \dot{H} = -4\pi G_N \sum_i (1+w_i)\rho_i + \frac{k}{a^2}, \tag{3.11}$$

where

$$\varrho(t) = \sum_i \rho_i(t), \quad \rho_i(t) = \rho_{i,0}\left(\frac{a(t)}{a_0}\right)^{-3(1+w_i)}. \tag{3.12}$$

The constraint can be written as

$$k = (aH)^2\left(\frac{8\pi G_N}{3H^2}\varrho - 1\right). \tag{3.13}$$

Notice that aH has dimensions inverse length and $\frac{8\pi G_N}{3H^2}$ dimensions of inverse energy density. For the later we define the "critical" energy density

$$\varrho_c(t) = \frac{3H^2(t)}{8\pi G_N}, \tag{3.14}$$

and thus

$$k = (a(t)H(t))^2\left(\frac{\varrho(t)}{\varrho_c(t)} - 1\right). \tag{3.15}$$

Then, the co-moving Hubble radius is defined as $(a(t)H(t))^{-1}$ and in order to make contact with observations the dimensionless density parameter is defined as

$$\Omega(t) = \frac{\varrho(t)}{\varrho_c(t)}. \tag{3.16}$$

The constraint takes the form

$$k = \left(\frac{1}{(a(t)H(t))^{-1}}\right)^2 (\Omega(t) - 1). \tag{3.17}$$

Notice that the left hand side is time independent. Then, for any two times t_0 and t_1, we will have

$$\left(\frac{1}{(a_1H_1)^{-1}}\right)^2 (\Omega_1 - 1) = \left(\frac{1}{(a_0H_0)^{-1}}\right)^2 (\Omega_0 - 1). \tag{3.18}$$

This is just a consequence that the topology does not change. Suppose that that Ω_0 is measured. This corresponds that we know H_0^2 and $\varrho_0 = \sum_i \rho_{i,0}$[2]. Then, there must be a tuning of parameters at t_1.

Now, there is a scenario in which today we can have $\Omega_0 = 1$ or more realistic $\Omega_0 \approx 1$. This means that $\varrho_0 \approx \varrho_{c0}$ and $k \approx 0$. Then, we would have

$$\left(\frac{1}{(a_1H_1)^{-1}}\right)^2 (\Omega_1 - 1) \approx 0. \tag{3.19}$$

On the other hand, we must solve

$$\dot{H} \approx -4\pi G_N \sum_i (1 + w_i)\rho_i. \tag{3.20}$$

In order to do so let us consider the ansatz

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^\beta. \tag{3.21}$$

Then the equation reduces to

$$\beta \left(\frac{t}{t_0}\right)^{-2} \approx \sum_i c_i \left(\frac{t}{t_0}\right)^{-3\beta(1+w_i)}, \tag{3.22}$$

where

$$c_i = \frac{4\pi G_N}{3}(1 + w_i)\rho_{i,0}t_0^2. \tag{3.23}$$

[2]Due to scale invariance in the spatial part of the metric, we can always set $a_0 = 1$. Here, will keep the factor a_0 with the understanding that is not a parameter to be measured. On the other hand knowing these quantities is not a trivial task. Apart from the actual measurements it must define what kind of fluids are in the universe.

Let us consider tree type of fluids: dust ($w_d = 0$), radiation ($w_r = 1/3$) and a strange fluid with ($w_v = -1$). Then,

$$\beta \left(\frac{t}{t_0}\right)^{-2} \approx c_d \left(\frac{t}{t_0}\right)^{-3\beta} + c_r \left(\frac{t}{t_0}\right)^{-4\beta}. \tag{3.24}$$

We see that the ansatz is suitable for $t < t_0$. For $t \ll t_0$ radiation dominates and thus $\beta = 1/2$. Hence, for $t \ll t_0$ we have that $\varrho(t) \approx \rho_r(t)$ and

$$a(t) \approx a_0 \left(\frac{t}{t_0}\right)^{\frac{1}{2}}, \quad H(t) \approx \frac{1}{2t_0}\left(\frac{t}{t_0}\right)^{-1}, \quad (a(t)H(t))^{-1} \approx \frac{2t_0}{a_0}\left(\frac{t}{t_0}\right)^{\frac{1}{2}}. \tag{3.25}$$

Notice that this universe faces a singularity in the past at $t = 0$. Thus, the solution can be trusted for $t \geq t_p$ where t_p is Planck's time. For $t_1 = t_p$, we obtain

$$\frac{1}{t_p}(\Omega_p - 1) \approx 0. \tag{3.26}$$

Since $t_p \sim 5.39 \times 10^{-44}$ this imply that $|\Omega_p - 1| \sim 10^{-44-\chi}$ with $\chi > 0$. This is an extreme tuning!

On the other hand, notice that at early stages the strange fluid does not contribute but for $t > t_0$ we see from (3.24) that it will be the only source. Then, for $w = -1$ the density is constant and thus $\varrho \approx \rho_{sf,0}$, where sf denotes strange and is a positive constant. Assume that now we start with

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 8\pi G_N T_{\mu\nu}. \tag{3.27}$$

Then, we interpret the cosmological constant as a source, i.e.

$$T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G_N} g_{\mu\nu}. \tag{3.28}$$

The equation result

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N (T_{\mu\nu} + T^{\Lambda}_{\mu\nu}), \tag{3.29}$$

where $T_{\mu\nu}$ includes dust and radiation. The cosmological constant as a fluid will give that

$$\rho_\Lambda = \frac{\Lambda}{8\pi G_N}. \tag{3.30}$$

If the strange fluid is the cosmological constant, then $\Lambda > 0$. The constraint for $t > t_0$ takes the form

$$H^2 \approx \frac{\Lambda}{3}, \tag{3.31}$$

and since the universe is expanding we obtain $a \sim e^{+\sqrt{\Lambda/3}t}$, i.e. it is exponentially expanding.

The whole scenario exposed above corresponds to out current model of our universe called the ΛCDM model[3]. In this model, dust stands for ordinary matter and

[3] See arXiv:2105.05208 for a detail discussion of this model.

dark matter. The strange fluid adjudicated to the cosmological constant Λ is referred as dark energy . Hence, its future behavior resembles a de Sitter spacetime with large ℓ, compare with equation (2.49). In summary, ignoring the singularity at $t = 0$, we have

$$a(t) \sim \begin{cases} e^{+\sqrt{\Lambda/3}t} & t > t_0 \quad \text{(Dark energy domination)} \\ t^{2/3} & t < t_0 \quad \text{(Matter domination)} \\ t^2 & t \ll t_0 \quad \text{(Radiation domination)} \end{cases} . \tag{3.32}$$

In the far future, the universe will be a very lonely place. A cartoon of the ΛCDM model is depicted in figure 3.1. Notice that for FRW universe, space is completely

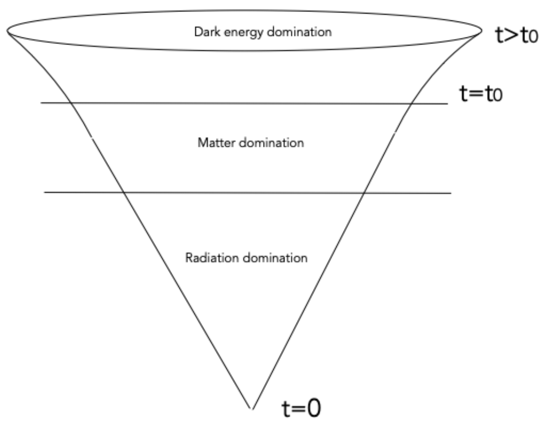

Figure 3.1: ΛCDM universe

homogenous and isotropic, this is because the spatial line segment is $\mathbb{R}^3$. Strictly speaking, this would mean that there would not be galaxies, star, planets, etc as we have measured. So we spect that at large enough scales, the universe should become statistically homogeneous and statistically isotropic[4]. In the boundary of radiation domination and matter domination (photons decouple), the cosmic microwave background (CMB) was created and gives a snapshot of the universe at that stage. The incredible measures of CMB indicates that is almost perfectly isotropic but with small anisotropies in its temperature[5]. These fluctuations, at linear order, are well described by a Gaussian random process. This is nice since it fits with the (classical) statistical mechanical notion of fluids and the assumption that the equations are sourced

[4]Notice that statistics is involved in two ways: as the usual way in an experiment or data analysis and in a (classical) statistical mechanical way.

[5]See arXiv:1303.5083.

by them. Moreover, these anisotropies are thought to be related with small perturbations, around the FRW background, that corresponds to the seed that forms the large scale structure of the universe[6]. However, there is no fundamental explanations of the Gaussian profile and its origin. The ΛCDM model is consistent with current observations but at the same time indicates the existence of dark energy and matter. Our current knowledge, i.e. the standard model of physics, has not yet establish the existence of "dark" particles[7]. On the other hand, dark energy has interpreted as the cosmological constant but no dynamical origin has been observed[8].

Returning to the flatness extreme tuning, notice that de Sitter with large ℓ also suggest that maybe there is a dynamically possibility in which $k \approx 0$ by a mechanism before radiation dominance. From

$$\left(\frac{1}{(a_1H_1)^{-1}}\right)^2(\Omega_1 - 1) = \left(\frac{1}{(a_0H_0)^{-1}}\right)^2(\Omega_0 - 1), \tag{3.33}$$

this would imply that there is an epoch for which the left hand side vanishes and thus $\Omega_0 \approx 1$ which is in agreement with observations. No tuning required. Of course the mechanism must end in order to transition to the radiation dominance epoch. The fluid that produces this mechanism must then convert to radiation.

This fluid is assumed to have a equation of state as $\rho = wp$ and thus $\rho = \rho_* a^{-3(1+w)}$. Since

$$k = (aH^2)(\Omega - 1), \tag{3.34}$$

we demand that $\Omega = 1$, so $\rho = \rho_c$. From the definition of ρ_c, we get

$$H^2 = \frac{8\pi G_N}{3}\rho. \tag{3.35}$$

We also have

$$\dot{H} = -4\pi G_N(1+w)\rho. \tag{3.36}$$

From the above expressions we obtain

$$-\frac{\dot{H}}{H^2} = \frac{3}{2}(1+w) \equiv \epsilon, \tag{3.37}$$

where ϵ codifies the nature of the fluid. The first expression combined with $\rho = \rho_* a^{-3(1+w)}$ gives

$$\dot{a}^2 = \frac{8\pi G_N}{3}\rho_* a^{2-2\epsilon}. \tag{3.38}$$

Taking a derivative we get

$$\ddot{a} = \frac{8\pi G_N}{3}(1-\epsilon)\rho_* a^{1-2\epsilon}. \tag{3.39}$$

[6] This subject is called: cosmological perturbation theory. See e.g. arXiv:hep-th/0306071v1.
[7] For a current review see arXiv:2104.11488.
[8] For a review see arXiv:1209.0922.

Again, the de Sitter lesson with large ℓ is that this epoch must satisfy $\ddot{a} > 0$. This epoch is called inflation[9].The condition $\ddot{a} > 0$ imply $\epsilon < 1$ or equivalently $w < -1/3$. Notice that for $\epsilon = 0$, i.e. $w = -1$, we again recover de Sitter. Without any surprise we consider the inflaton field ϕ. We already now that the potential must have regions that is nearly flat (de Sitter approximation). But now we also now that inflation must end ($\epsilon \to 1$) and thus the potential must go (slow roll) to zero. Around the zero potential we have a transition to the universe dominated by radiation. A prototypical potential with slow roll is sketched in figure 3.2.

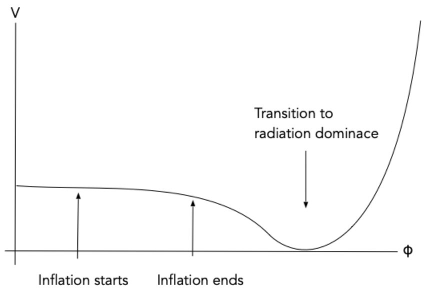

Figure 3.2: Slow roll potential

For these type of potentials we see that when inflation starts the potential term dominates and when it ends the kinetic term dominates. Then, the kinetic energy of the inflation converts to give the radiation fluid.

This is strange from the point of view of the no-boundary proposal. We expect that inflation should start from the bottom of the well not on the plateau. Moreover, we need a lot of inflation to produce the current structure in the universe[10].

Hence, we see that Ψ_{HH} is not naturally compatible with slow-roll inflation[11].

However, inflation models faces phenomenological and theoretical issues. The slow roll inflation predicts small non-Gaussianities and thus more accurate measurements of the CMB will be required to validate the theory[12]. On the theoretical side, inflation moves the ΛCDM singularity back into an indefinite past[13]. It is only it is a

[9] For a more detail motivation of inflation and its consequences see arXiv:0907.5424. One of the outstanding features of inflation is that as a semiclassical quantum theory, the origin of the seeds of the cosmological perturbations that would lead to the large scale structure of the universe are completely quantum.

[10] For details on the physics, see arXiv:0811.3919.

[11] There is a particular solution of this issue, it demands to take into account the presence of the observer in the conditional probability. It is a solution coming from the interpretation of quantum mechanics applied to the universe rather than a dynamical solution. See arXiv:1503.07205 , "Volume Weighting in the No Boundary Proposal" arXiv:0710.2029v1 by Hawking and "The no-boundary measure of the universe", Phys. Rev. Lett. 100, 202301 (2008), arXiv:0711:4630, by Hartle, Hawking and Hertog.

[12] See Planck 2018 results. X. Constraints on inflation.

[13] See arXiv:gr-qc/0110012 and arXiv:gr-qc/9612036v1.

reasonable semiclassical quantum theory but tells us nothing about the quantum origin of the universe. Ψ_{HH} does and therefore there is a possibility that the potential used in slow roll inflation is too naive. Moreover, the specific saddle that produces Ψ_{HH} give the same prediction of tensor perturbations as in inflation[14].

Notice that slow-roll inflation seems to prefer (tunneling) solutions as Ψ_B[15]. But in our interpretation of probability, this type of universe has low likelihood to exist since ℓ has to be large.

For a review of the no-boundary proposal (and its issue with slow roll inflation) see the latest comments by Maldacena in arXiv:2403.10510v1 and references within.

I end this note indicating that if a dynamical solution to the incompatibility of the no-boundary proposal with slow roll inflation is given, the (classical) big bang singularity will not exist.

[14] See arXiv:2303.08802.

[15] The solution is plotted in figure 2.2, we see that it resembles a wave function that tunnels a barrier. In this case it corresponds to the part of the potential U_{eff} in the no-classical region, see figure 2.1 and recall that this region corresponds to $-1 \leq \zeta \ll 0$. For Vilenkin's tunneling proposal see arXiv:1808.02032v2 and specially "Quantum cosmology and the initial state of the Universe" Phys. Rev. D 37, 888 1988.

Appendix A

Conditional probabilities

Consider an experiment that has two possibles outcomes: $\uparrow$ and $\downarrow$. Then, the sample space is $\Omega = \{\uparrow, \downarrow\}$. The possible events, the subsets of the sample space, are $\{\uparrow, \downarrow\}$, $\{\uparrow\}$, $\{\downarrow\}$ and $\{\emptyset\}$. The cardinality of the set $\{\uparrow, \downarrow\}$ is 2 and for the remaining is 1. Then, the probability of the sample is space is defined to be

$$P(\Omega) = 1. \tag{A.1}$$

This is the normalization axiom.

Since $\Omega = \{\uparrow\} \cup \{\downarrow\}$, which means that $\{\uparrow\} \cap \{\downarrow\} = \{\emptyset\}$, the additivity axiom states that $P(\Omega) = P(\{\uparrow\}) + P(\{\downarrow\})$ and therefore $P(\{\uparrow\}) + P(\{\downarrow\}) = 1$. Notice that if we relax the normalization axiom, the additivity axiom will tell us that

$$\frac{P(\{\uparrow\})}{P(\{\uparrow, \downarrow\})} + \frac{P(\{\downarrow\})}{P(\{\uparrow, \downarrow\})} = 1. \tag{A.2}$$

Now, notice that

$$\{\uparrow\} = \{\uparrow\} \cap \{\uparrow, \downarrow\}, \quad \{\downarrow\} = \{\downarrow\} \cap \{\uparrow, \downarrow\}. \tag{A.3}$$

Then, we can write

$$\frac{P(\{\uparrow\} \cap \{\uparrow, \downarrow\},)}{P(\{\uparrow, \downarrow\})} + \frac{P(\{\downarrow\} \cap \{\uparrow, \downarrow\})}{P(\{\uparrow, \downarrow\})} = 1. \tag{A.4}$$

Consider the classical scattering problem of sending a stream of particles towards a thin foil, see figure A.1. Detectors are place in both sides of the plate. The experiment register that some particles are reflected and some pass through the plate. Classically we idealize the physical problem by assuming that the detectors are located at the same distance L and that the particles are thought as tiny balls with radius r_p. Also we assume that r_p is significantly smaller than L. On the other hand, we will only consider detections in a narrow region along the particle beam. Ignore how the

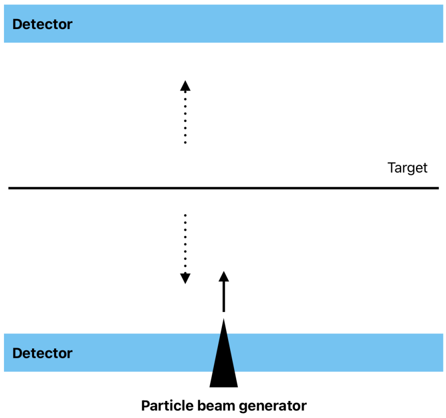

Figure A.1: Scattering problem. The target is a thin foil.

detector is placed in the same spot of the particle beam generator. Then, the problem becomes a one dimensional problem. Mathematically, the assumptions allow us to place the detectors at $x \to -\infty$ and $x \to +\infty$. Therefore, the target is located around $x = 0$. On the other hand, the stream of particles are assumed to carry the same momentum p_0.

Let us call $\{\uparrow\}$ the event of the particles passing through the target and $\{\downarrow\}$ the event of particles reflected. We then can consider the case in which some particles of the incoming beam pass through and other are reflected. Thus, $P(\{\uparrow,\downarrow\})$ is assume to exist but we do not know its value. Then, $\frac{P(\{\uparrow\}\cap\{\uparrow,\downarrow\})}{P(\{\uparrow,\downarrow\})}$ is interpreted as the probability that particles pass through given that both detectors record a signal and $\frac{P(\{\downarrow\}\cap\{\uparrow,\downarrow\})}{P(\{\uparrow,\downarrow\})}$ as the probability that particles pass through given that both detectors record a signal. This are conditional probabilities. Then, we see that the sum of conditional probabilities, do to the additive axiom, is one.

Now consider the quantum interpretation of the scattering problem. Now consider the de Broglie wavelength of the particles λ_p instead of r_p. Again $L \gg \lambda_p$. We also consider scattering states, which are not normalizable, and the momentum of each particle is k_0. The target is the compact potential. The coefficient of the incoming plane wave is c and the coefficients of the reflected and transmitted plane waves are $r(k_0)$ and $t(k_0)$ respectively. Conservation of the probability flux gives

$$\frac{|r(k_0)|^2}{|c|^2} + \frac{|t(k_0)|^2}{|c|^2} = 1. \tag{A.5}$$

Then we consider

$$P(\{\uparrow,\downarrow\}) = |c|^2, \quad P(\{\downarrow\}\cap\{\uparrow,\downarrow\}) = |r(k_0)|^2, \quad P(\{\uparrow\}\cap\{\uparrow,\downarrow\}) = |t(k_0)|^2. \tag{A.6}$$

Hence, $|c|^2$ corresponds to the probability that after sending the incoming plane wave, there will be a reflected plane wave and a transmitted plane wave. Then, $P_r = \frac{|r(k_0)|^2}{|c|^2}$, $P_t = \frac{|t(k_0)|^2}{|c|^2}$ are indeed conditional probabilities.

Appendix B

Ground state wave function of the harmonic oscillator

For a one-dimensional quantum problem, consider the probability amplitude $\langle x_2, t_2 | x_1, t_1 \rangle$. Using the evolution operator $\hat{U}(t, t_1) = \mathrm{e}^{-\frac{i(t-t_1)}{\hbar}\hat{H}}$.

$$\langle x_2, t_2 | x_1, t_1 \rangle = K(x_2, t_2, x_1, t_1) = \sum_{n=0} \psi_n(x_2)\psi_n^*(x_1)\mathrm{e}^{-i(t_2-t_1)\frac{E_n}{\hbar}}, \tag{B.1}$$

where $\hat{H}\psi_n = E_n\psi_n$. The propagator K has a path integral representation

$$K(x_2, t_2, x_1, t_1) = \int_{x(t_1)=x_1}^{x(t_2)=x_2} \mathcal{D}x(t)\, \mathrm{e}^{\frac{i}{\hbar}S[x(t)]}, \tag{B.2}$$

with

$$S[x(t)] = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{m}{2} \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 - V(x(t)) \right). \tag{B.3}$$

Then, the amplitude can be written as

$$\langle x_2, t_2 | x_1, t_1 \rangle = \int_{x(t_1)=x_1}^{x(t_2)=x_2} \mathcal{D}x(t)\, \mathrm{e}^{\frac{i}{\hbar}S[x(t)]}. \tag{B.4}$$

Consider now the euclidean version by taking $t = -it_E$, then

$$\langle x_2, t_{E2} | x_1, t_{E1} \rangle = \sum_{n=0} \psi_n(x_2)\psi_n^*(x_1)\mathrm{e}^{-(t_{E2}-t_{E1})\frac{E_n}{\hbar}}. \tag{B.5}$$

Let $x_1 = 0$ and $t_{E2} = 0$, we get

$$\langle x_2, 0 | 0, t_{E1} \rangle = \sum_{n=0} \psi_n(x_2)\psi_n^*(0)\mathrm{e}^{t_{E1}\frac{E_n}{\hbar}}, \tag{B.6}$$

and thus, the ground state $\psi_0(x_2)$ can be computed from

$$\psi_0(x_2) \propto \int\limits_{x(-\infty)=0}^{x(0)=x_2} \mathcal{D}x(t_E)\, \mathrm{e}^{-\frac{1}{\hbar}S_E[x(t_E)]}, \tag{B.7}$$

where

$$S_E[x(t_E)] = \int\limits_{-\infty}^{0} \mathrm{d}t_E \left(\frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t_E}\right)^2 + V(x(t_E))\right). \tag{B.8}$$

Evaluating functional integrals (i.e. path integrals) is hard and we know how to compute them only if the Lagrangian is quadratic in its variables. For this reason let us consider the euclidean harmonic oscillator

$$L_E = \frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t_E}\right)^2 + \frac{m\omega^2}{2}x^2. \tag{B.9}$$

Notice that the potential is inverted and now we have an unstable maximum. The action can be written as

$$S_E[x(t_E)] = \frac{m}{2}x(0)\frac{\mathrm{d}x}{\mathrm{d}t_E}(0) - \frac{m}{2}x(-\infty)\frac{\mathrm{d}x}{\mathrm{d}t_E}(-\infty) + \frac{m}{2}\int\limits_{-\infty}^{0} \mathrm{d}t_E\, x\hat{O}x, \tag{B.10}$$

where

$$\hat{O} = -\frac{\mathrm{d}^2}{\mathrm{d}t_E^2} + \omega^2 \tag{B.11}$$

The classical equation of motion is $\frac{\mathrm{d}^2 x_c}{\mathrm{d}t_E^2} = \omega^2 x_c$ and the solution is of the form $x_c(t_E) = A\mathrm{e}^{\omega t_E} + B\mathrm{e}^{-\omega t_E}$. Since the path integral demands that $x(-\infty) = 0$, $x(0) = x_2$ we find that the classical solution is $x_c(t_E) = x_2\mathrm{e}^{\omega t_E}$. Since $\hat{O}x_c = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t_E}(0) = \omega x_2$ and $\frac{\mathrm{d}x_c}{\mathrm{d}t_E}(-\infty) = 0$, the on-shell action gives

$$S_E[x_c] = \frac{m\omega}{2\hbar}x_2^2. \tag{B.12}$$

and moreover the factor

$$\mathrm{e}^{-\frac{1}{\hbar}S_E[x_c]} = \mathrm{e}^{-\frac{m\omega}{2\hbar}x_2^2}, \tag{B.13}$$

reproduces the ground state wave function. This suggest that the path integral should be computed by assuming

$$x(t_E) = x_c(t_E) + \hbar\eta(t_E), \tag{B.14}$$

where η corresponds to the quantum fluctuation around the classical path x_c. These fluctuations must obey $\eta(0) = 0 = \eta(-\infty)$. Then

$$S_E[x_c+\eta] = S_E[x_c] + \hbar\left[m\left(\eta\frac{\mathrm{d}x_c}{\mathrm{d}t_E}\right)\Bigg|_{t\to-\infty}^{0} - \int\limits_{-\infty}^{0} m\eta\left(\frac{\mathrm{d}^2 x_c}{\mathrm{d}t_E^2} - \omega^2 x_c\right)\right] + \hbar^2 S_E[\eta]. \tag{B.15}$$

and therefore

$$\int_{x(-\infty)=0}^{x(0)=x_2} \mathcal{D}x(t_E)\, \mathrm{e}^{-\frac{1}{\hbar}S_E[x(t_E)]} = \mathrm{e}^{-\frac{1}{\hbar}S_E[x_c]} \int_{\eta(-\infty)=0}^{\eta(0)=0} \mathcal{D}\eta(t_E)\, \mathrm{e}^{-\hbar S_E[\eta]}. \quad \text{(B.16)}$$

For the harmonic oscillator we see that the remaining path integral must give a constant. Beyond the harmonic oscillator, the semiclassical ground state can be computed from the integral by taking $\hbar \to 0$. Thus we find

$$\psi_0^{\text{semiclassical}}(x_2) \sim \sum_i \mathrm{e}^{-\frac{1}{\hbar}S_E[x_c^i]}, \quad \text{(B.17)}$$

where the sum is taken for all possible euclidean classical solutions that satisfy the boundary data. Notice that the right hand side of

$$\psi_0(x_2) \propto \int_{x(-\infty)=0}^{x(0)=x_2} \mathcal{D}x(t_E)\, \mathrm{e}^{-\frac{1}{\hbar}S_E[x(t_E)]}, \quad \text{(B.18)}$$

is viewed as a mathematical technique to find the ground state. The physical input is $x(0) = x_2$ and the action S. The euclidean time and action, together with the boundary data corresponds to a prescription not a areal physical system.

In order to enforce this interpretation, notice that after fixing x_1, t_1 the propagator K satisfy Schrödinger's equation. By splitting $x(t) = x_c(t) + \hbar\eta$ with $\eta(t_1) = 0 = \eta(t_2)$ we find that an action with potential $V(x)$ result

$$S[x_c + \hbar\eta] = S[x_c] + \hbar^2 I[\eta, \hbar), \quad \text{(B.19)}$$

with

$$I[\eta, \hbar) = \int_{t_1}^{t_2} \mathrm{d}t\, \left(\frac{m}{2}\dot{\eta}^2 - \frac{1}{2!}V''\eta^2 - \frac{\hbar}{3!}V'''\eta^3 - \frac{\hbar^2}{4!}V''''\eta^4 + \dots \right), \quad \text{(B.20)}$$

and therefore

$$K(x_2, t_2, x_1, t_1) = \mathcal{A}\, \mathrm{e}^{\frac{i}{\hbar}S[x_c]} \quad \mathcal{A} = \int_{\eta(t_1)=0}^{\eta(t_2)=0} \mathcal{D}\eta(t)\, \mathrm{e}^{i\hbar I[\eta,\hbar)}, \quad \text{(B.21)}$$

where $x_c(t_1) = x_1$ and $x_c(t_2) = x_2$. In the limit $\hbar \to 0$, we find

$$\psi(x_2, t_2) \sim \mathrm{e}^{\frac{i}{\hbar}S[x_c]}. \quad \text{(B.22)}$$

Notice that $x_c(t_1) = x_1$ is not provided but must be fixed. Let us consider again the harmonic oscillator. We have shown that for a classical solution of the form

$$x_c(t) = A\cos(\omega t) + B\sin(\omega t), \quad \text{(B.23)}$$

we have

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{\sin(\omega t_2)}{\sin(\omega(t_2-t_1))}x_1 - \frac{\sin(\omega t_1)}{\sin(\omega(t_2-t_1))}x_2 \\ -\frac{\cos(\omega t_2)}{\sin(\omega(t_2-t_1))}x_1 + \frac{\cos(\omega t_1)}{\sin(\omega(t_2-t_1))}x_2 \end{pmatrix}. \tag{B.24}$$

Let $x_1 = 0$, then

$$x_c(t) = x_2 \frac{\sin(\omega(t-t_1))}{\sin(\omega(t_2-t_1))}, \quad t_1 \leq t \leq t_2. \tag{B.25}$$

The on-shell action result $S[x_c] = \frac{m\omega x_2^2}{2}\cot(\omega(t_2-t_1))$. Due to the singularities of the cotangent function we consider the analytical continuation $t = -it_E$ and thus we find

$$\cot(\omega(t_2-t_1)) = i\coth(\omega(t_{E2}-t_{E1})). \tag{B.26}$$

From the hyperbolic cotangent function, we see that it is reasonable to consider $t_{E2} = 0$ and $t_{E1} \to -\infty$. This will select automatically the ground state. With this perspective, we see that the analytical continuation is a method of regularization.

On the other hand, by setting x_2 to be a general point x at some arbitrary t, we find that the semiclassical wavefunction is of the form

$$\psi^{\text{semiclassical}}(x,t) \propto \mathrm{e}^{\frac{i}{\hbar}\mathcal{S}_E(x,t,x_1,t_1)}, \tag{B.27}$$

where $\mathcal{S}$ is the Hamiltons principal function defined as

$$\mathcal{S}(x,t,x_1,t_1) = \int_{t_1}^{t} \mathrm{d}u \left(\frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 - V(x)\right). \tag{B.28}$$

Hamiltons principal function satisfy Hamilton-Jacobi equation

$$-\frac{\partial \mathcal{S}}{\partial t} = H\left(x, \frac{\partial \mathcal{S}}{\partial x}, t\right). \tag{B.29}$$

Which has the form

$$-\frac{\partial \mathcal{S}}{\partial t} = \frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + V(x). \tag{B.30}$$

Let us assume that $\mathcal{S} = -E(t-t_1) + W(x) - W(x_1)$ and therefore

$$\psi^{\text{semiclassical}}(x,t) \propto \mathrm{e}^{-\frac{i}{\hbar}E(t-t_1)+\frac{i}{\hbar}(W(x)-W(x_1))}, \tag{B.31}$$

and

$$E = \frac{1}{2m}\left(W'\right)^2 + V(x). \tag{B.32}$$

From this we find

$$\Delta W = W(x) - W(x_1) = \pm \int_{x_1}^{x} \mathrm{d}x' \sqrt{2m(E-V(x'))}. \tag{B.33}$$

The classical allowed region corresponds to $E > V$ and the non- classical region to $E < V$. For the harmonic oscillator we get

$$W(x) - W(x_1) = \pm m\omega \int_{x_1}^{x} \mathrm{d}x' \sqrt{\ell^2 - x'^2}, \quad \ell^2 = \frac{2E}{m\omega^2}, \tag{B.34}$$

where $\pm\ell$ corresponds to the returning points. The classical region corresponds to $x^2 < \ell^2$ and the non- classical to $x^2 > \ell^2$. In the non- classical region we write

$$W(x) - W(x_1) = \pm i m\omega \int_{x_1}^{x} \mathrm{d}x' \sqrt{x'^2 - \ell^2}. \tag{B.35}$$

After performing the integral we obtain

$$W(x) - W(x_1) = \pm i m\omega \left[\frac{1}{2} x \sqrt{x^2 - \ell^2} - \frac{1}{2} x_1 \sqrt{x_1^2 - \ell^2} - \frac{\ell^2}{2} \ln \left(\frac{x + \sqrt{x^2 - \ell^2}}{x_1 + \sqrt{x_1^2 - \ell^2}} \right) \right]. \tag{B.36}$$

If we consider the positive solution, and set $\ell = 0$, $x_1 = 0$ we find

$$\psi_0^{\text{semiclassical}}(x,t) \propto \mathrm{e}^{-\frac{i}{\hbar} E(t - t_1)} \mathrm{e}^{-\frac{m\omega}{2\hbar} x^2}, \tag{B.37}$$

where the factor $E(t - t_1)$ survives only if $(t - t_1) \to \infty$.